MATH 8306: ALGEBRAIC TOPOLOGY
PROBLEM SET 1, DUE MONDAY, OCTOBER 4, 2004

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Section 2.3 (from Hatcher's textbook): 1, 2, 4 (Do #4 for unreduced homology and the wedge axiom replaced by the additivity axiom)
Appendix A, p. 529: 2, 3, 4

Problem 1. Show that a CW complex is path-connected iff it is connected.

Problem 2. Let X be a topological space and x_0, x_1 given points in it. Let Y be the space of paths starting at x_0 and *passing* through x_1. Show that Y is contractible. [Hint: This problem is not a misprint, although may seem wrong on the first sight. Use the compact-open topology on the space of paths, see Hatcher, p. 529.]

Problem 3. Represent a compact orientable surface of genus g (i.e., the sphere with g handles) as a CW complex.

Problem 4. Is the subset $\bigcup_{n=1}^{\infty}\{(x,y) \mid x^2 - x/n + y^2 = 0\}$ of $\mathbb{R}^2$ a CW complex for the cell structure given by $X^0 = \{(0,0)\}$ and the 1-cells being the complements of $(0,0)$ in the circles $x^2 - x/n + y^2 = 0$?

Problem 5. Show that the subspace $\{0, 1, 1/2, \dots, 1/n, \dots\} \subset \mathbb{R}$ does not have the homotopy type of a CW complex. [This means it is not homotopy equivalent to a CW complex.]

Problem 6. Derive the following unreduced version of the Mayer-Vietoris sequence, using the (generalized) homology axioms. If A and B are subcomplexes of a CW complex X and $X = A \cup B$, then the *Mayer-Vietoris sequence*

$$\cdots \to E_p(A \cap B) \xrightarrow{\phi} E_p(A) \oplus E_p(B) \xrightarrow{\psi} E_p(X) \xrightarrow{\partial_p} E_{p-1}(A \cap B) \to \dots,$$

where ϕ is the sum of the homomorphisms induced by the inclusions, ψ is the difference of those induced by the inclusions, and ∂ is a certain connecting homomorphism, is exact.

Problem 7. Use the Mayer-Vietoris sequence to give compute the homology groups of spheres.

Problem 8. Use the Mayer-Vietoris sequence to compute the homology of the space which is the union of three n-disks along their common boundaries.

Problem 9. Calculate the homology of $S^p \vee S^q$ using the Mayer-Vietoris sequence.

Problem 10. Let A be a subcomplex of a CW complex X, Y a CW complex, $f : A \to Y$ a cellular map, and $Y \cup_f X$ the pushout (i.e., X attached to Y along f). The *Euler characteristic* of a finite CW complex X may be defined as $\chi(X) =$

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$\sum_n (-1)^n \gamma_n(X)$, where γ_n is the number of n-cells of X. Formulate and prove a formula relating the Euler characteristics of A, X, Y, and $X \cup_f Y$, when X and Y are finite.

For a graded vector space $V = \bigoplus V_n$ with $V_n = 0$ for all but finitely many n and with all V_n finite dimensional, define the *Euler characteristic* $\chi(V)$ to be $\sum_n (-1)^n \dim V_n$.

Problem 11. Let V', V, and V'' be such graded vector space and suppose there is a long exact sequence

$$\cdots \to V'_n \to V_n \to V''_n \to V'_{n-1} \to \ldots$$

Prove that $\chi(V) = \chi(V') + \chi(V'')$.

Problem 12. If, for such a vector space V, $\{V_n, d_n\}$ is a chain complex, show that $\chi(V) = \chi(H_\bullet(V))$.

Problem 13. Let $p : X \to Y$ be a covering map with finite fibers of cardinality n. Using singular chains, construct a homomorphism $t : H_\bullet(Y; G) \to H_\bullet(X; G)$ such that the composite $p_* \circ t : H_\bullet(Y; G) \to H_\bullet(Y; G)$ is a multiplication by n. [t is called a *transfer homomorphism.*]