MATH 8306: ALGEBRAIC TOPOLOGY
PROBLEM SET 2, DUE WEDNESDAY, NOVEMBER 17, 2004

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Section 2.2 (from Hatcher's textbook): 1, 2 (omit the question about odd dimensional projective spaces)

Problem 1. Apply the method of acyclic models to show that the barycentric subdivision operator is homotopic to identity. Reminder: we used that method to prove that singular homology satisfies the homotopy axiom. It consisted in constructing a *natural* chain homotopy inductively for all spaces at once by using the acyclicity of the singular simplex, i.e., the vanishing of its higher homology. You may read more about this method in Selick's text, pp. 35–37.

Problem 2. Compute the simplicial homology of the Klein bottle.

Problem 3. Prove the boundary formula for cellular 1-chains:

$$d^{\mathrm{CW}}(e^1_\alpha) = [1_\alpha] - [-1_\alpha],$$

where e^1_α is a 1-cell of a CW complex and $[1_\alpha]$ and $[-1_\alpha]$ are its endpoints, identified with 0-cells of the CW complex via the attaching maps.

Problem 4. Using cell decompositions of S^n and D^n, compute their cellular homology groups.

Problem 5. Calculate the integral homology groups of the sphere M_g with g handles (*i.e.*, a compact orientable surface of genus g) using a cell decomposition of it.

Problem 6. Give a simple proof that uses homology theory for the following fact: $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for $m \neq n$.

Problem 7. Let $0 \to \pi \xrightarrow{f} \rho \xrightarrow{g} \sigma \to 0$ be an exact sequence of abelian groups and C a chain complex of *flat* (i.e., torsion free) abelian groups. Write $H_\bullet(C; \pi) = H_\bullet(C \otimes \pi)$. Construct a natural long exact sequence

$$\cdots \to H_q(C;\pi) \xrightarrow{f_*} H_q(C;\rho) \xrightarrow{g_*} H_q(C;\sigma) \xrightarrow{\beta} H_{q-1}(C;\pi) \to \dots.$$

The connecting homomorphism β is called a *Bockstein homomorphism*.

Problem 8. Let p be an odd prime number. Regard the cyclic group $\mathbb{Z}_p$ of order p as the group of pth roots of unity contained in S^1. Regard S^{2n-1} as the unit sphere in $\mathbb{C}^n$, $n \geq 2$. Then $\mathbb{Z}_p$ acts freely on S^{2n-1} via

$$\zeta(z_1, \dots, z_n) = (\zeta z_1, \dots, \zeta z_n).$$

Let $L^n = S^{2n-1}/\mathbb{Z}_p$ be the orbit space; it is called a *lens space* and is an odd primary analogue of $\mathbb{RP}^n$. The obvious quotient map $S^{2n-1} \to L^n$ is a universal covering.

Date: November 1, 2004.

(1) Compute the integral homology of L^n, $n \geq 2$, by mimicking the calculation of $H_\bullet(\mathbb{RP}^n)$.
(2) Compute $H_\bullet(L^n; \mathbb{Z}_p)$.

Problem 9. Use the integral homology of the real projective plane $\mathbb{RP}^2$,

$$H_n(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}_2 & \text{for } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

to compute the homology $H_\bullet(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$ of the product space $\mathbb{RP}^2 \times \mathbb{RP}^2$.

Problem 10. Compute the homology of the Klein bottle over $\mathbb{Z}_2$ using its integral homology groups.