MATH 8306: ALGEBRAIC TOPOLOGY
PROBLEM SET 3, DUE FRIDAY, DECEMBER 17, 2004

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Problem 1. Use the Universal Coefficient theorem and the Künneth formula for homology to show that for a field R, we have a natural isomorphism:

$$\alpha : H^\bullet(X; R) \otimes_R H^\bullet(Y; R) \to H^\bullet(X \times Y; R),$$

where X and Y are CW complexes, $H^\bullet(-; R)$ is the cellular cohomology with coefficients in R, and at least one homology space, say, $H_\bullet(Y; R)$ is assumed finite-dimensional. [You may skip checking the naturality: we constructed the above homomorphism in class, called it the cross product, and explained why it was natural. The only thing to prove is that it is an isomorphism.]

Problem 2. Let X and Y be CW complexes. Show that the interchange map

$$\tau : X \times Y \to Y \times X$$

satisfies $\tau_*([i] \otimes [j]) = (-1)^{pq}[j] \otimes [i]$ for a p-cell $[i]$ of X and a q-cell $[j]$ of Y. Deduce that the cohomology ring $H^\bullet(X)$ is graded commutative, i.e.,

$$x \cup y = (-1)^{pq} y \cup x, \quad \text{if } \deg x = p \text{ and } \deg y = q.$$

Problem 3. Let X be an $(n-1)$-connected (here: $\tilde{H}^q(X) = 0$ for $q < n$) H-space X (see Section 3.C of Hatcher, except that we do not make that extra assumption about the cross products and Hatcher's Δ is our μ^*, while here $\Delta : X \to X \times X$ is the diagonal map) and $x \in H^n(X)$.

(1) Show that $\mu^*(x) = x \otimes 1 + 1 \otimes x$.
(2) Show that $(\Delta \times \Delta)^*(\mathrm{id} \times \tau \times \mathrm{id})^*(\mu \times \mu)^*(x \otimes x) = x^2 \otimes 1 + (1 + (-1)^n)(x \otimes x) + 1 \otimes x^2$. [Hint: Observe that the composition in the previous part at the level of spaces is equal to $\Delta\mu : X \times X \to X \times X$ and recall how the cup product is related to the diagonal. The property "$\mu^*\Delta^*$ equals the composition on the left-hand side of the formula you are asked to prove" implies that $H^\bullet(X)$ is a *Hopf algebra*, when you use coefficients in a field.]
(3) Prove that, if n is even, then either $2(x \otimes x) = 0$ in $H^\bullet(X \times X)$ or $x^2 \neq 0$. Deduce that S^n cannot be an H-space, if n is even.

Problem 4. Assuming as known the cup product structure on the torus $T^2 = S^1 \times S^1$ ($(e^0)^* \cup a = a$ for a generator $(e^0)^*$ of $H^0(T; \mathbb{Z}) = \mathbb{Z}$ and any $a \in H^\bullet(T; \mathbb{Z})$, $(e^1_1)^* \cup (e^1_2)^* = (e^2)^*$ and $(e^1_i)^* \cup (e^1_i)^* = 0$ for $i = 1, 2$ and generators $(e^1_1)^*$ and $(e^1_2)^*$ of $H^1(T; \mathbb{Z})$ coming from two natural projections $T \to S^1$ via pullback, and $(e^2)^* \cup (e^1_i)^* = (e^2)^* \cup (e^2)^* = 0$), compute the cup product on $H^\bullet(M_g; \mathbb{Z})$ for the compact orientable surface M_g of genus g, by using a quotient map from M_g to the bouquet of g tori.

Date: November 24, 2004, updated Dec 10 and 13.

Problem 5. Show that any map $S^4 \to S^2 \times S^2$ must induce the zero homomorphism on $H_4(-)$. [Hint: Use the cup product]

Problem 6. Prove that there is no homotopy equivalence $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ that reverses orientation (i.e., induces multiplication by -1 on $H_{4n}(\mathbb{CP}^{2n};\mathbb{Z})$). [Hint: note from cellular cohomology that the generator $(e^{4n})^* \in H^{4n}$ is a cup square $(e^{2n})^* \cup (e^{2n})^*$, which you may assume is true.]

Problem 7. Let $X = \Sigma Y = Y \wedge S^1$ be a reduced suspension. Show that the cup product $\tilde{H}^p(X) \otimes \tilde{H}^q(X) \to \tilde{H}^{p+q}(X)$ is the zero homomorphism, where the coefficients are assumed to be in a commutative ring R and the tensor product is over R. [Hint: For a pointed space X and two open subspaces A and B with a basepoint $* \in A \cap B$, construct a commutative diagram

$$\begin{array}{ccc} H^p(X,A) \otimes H^q(X,B) & \longrightarrow & H^{p+q}(X, A \cup B) \\ \downarrow & & \downarrow \\ \tilde{H}^p(X) \otimes \tilde{H}^q(X) & \longrightarrow & \tilde{H}^{p+q}(X) \end{array}$$

and use it in the case $X = A \cup B$, where A and B are contractible.]

Problem 8. Let M be a compact orientable n-manifold. Suppose that M is homotopy equivalent to ΣY for some connected pointed space Y. Deduce that M has the same integral homology as S^n. [Comment: amazingly enough, this implies that M is homotopy equivalent to S^n via the Whitehead theorem, which concludes that a map between CW complexes inducing an isomorphism on homology is a homotopy equivalence.]

Problem 9. Prove that for $n \leq \infty$, $H^\bullet(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[t]/(t^{n+1})$ as a ring, with degree $|t| = 1$. [Hint: This is a repetition of what we (will) have done in class for $\mathbb{CP}^n$, plus a new statement for $n = \infty$, where Poincaré duality does not work directly.]

In the problems below, M is assumed to be a compact connected n-manifold (without boundary) and $n \geq 2$.

Problem 10. Prove that if M is a Lie group, then M is orientable.

Problem 11. Prove that if M is orientable, then $H_{n-1}(M;\mathbb{Z})$ is a free abelian group.