MATH 8307: ALGEBRAIC TOPOLOGY
PROBLEM SET 5, DUE FRIDAY, APRIL 15, 2005

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From Hatcher's textbook:
Section 4.1: 18
Section 4.2 (from Hatcher's textbook): 15 (In fact, prove a more general statement that a closed connected simply-connected n-dimensional manifold whose homology is the same as that of S^n is homotopy equivalent to S^n.) [Hint: use the Hurewicz theorem.]

Problem 1. Show that $\pi_7(S^4)$ contains a $\mathbb{Z}$ summand.

Problem 2. Compute all the homotopy groups of $\mathbb{R}P^\infty = \bigcup_{n\geq 1} \mathbb{R}P^n$, using the computation of the homotopy groups of $\mathbb{R}P^n$ through the homotopy groups of spheres S^n.

Problem 3. Regarding a singular cochain $\phi \in C^1(X; G)$ as a function from paths in X to G, show that if ϕ is a cocycle, that is, $\delta\phi = 0$, then

(1) $\phi(f \cdot g) = \phi(f) + \phi(g)$,
(2) ϕ takes the value 0 on constant paths,
(3) $\phi(f) = \phi(g)$ if $f \sim g$ via a homotopy fixing the endpoints,
(4) ϕ is a coboundary (that is, $\phi = \delta\psi$ for some $\psi \in C^0(X; G)$) iff $\phi(f)$ depends only on the endpoints of f, for all f.

[In particular, 1 and 4 give a homomorphism $H^1(X; G) \to \mathrm{Hom}(\pi_1(X), G)$, which is a version of Hurewicz isomorphism if X is path connected.]

Problem 4. Show that, if $n \geq 2$, then $\pi_n(X \vee Y)$ is isomorphic to

$$\pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

Problem 5. Compute $\pi_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ for $n \geq 2$. Deduce that the quotient map

$$(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \to (\mathbb{RP}^n/\mathbb{RP}^{n-1}, *)$$

does not induce an isomorphism of homotopy groups.

Problem 6. Assume given maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$. Suppose that Y is a CW complex. Show that X has the homotopy type of a CW complex, *i.e.*, is homotopy equivalent to a CW complex.

Problem 7. Let $n \geq 1$ and π be an abelian group. Construct a connected CW complex X such that $\widetilde{H}_n(X; \mathbb{Z}) = \pi$ and $\widetilde{H}_q(X; \mathbb{Z}) = 0$ for $q \neq n$. Such space X is denoted $M(\pi, n)$ and called a *Moore space*. [Hint: construct $M(\pi, n)$ as the cofiber of a map between wedges of spheres.]

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Problem 8. Let $n \geq 1$ and π be an abelian group. Construct a connected CW complex X such that $\pi_n(X) = \pi$ and $\pi_q(X) = 0$ for $q \neq n$. Such space X is denoted $K(\pi, n)$ and called an *Eilenberg-Mac Lane space*. [Hint: start with $M(\pi, n)$, use the Hurewicz theorem, and kill the higher homotopy groups.]