LECTURE 10: MODULI SPACES OF ALGEBRAIC CURVES OF GENUS ZERO

ALEXANDER A. VORONOV

1. Moduli spaces of algebraic curves of genus zero

The moduli spaces of algebraic curves have been a very exciting object to study in algebraic geometry. Perhaps, it is so, because very little is known about it, while on the other hand, so much is known about algebraic curves, which form the points of the moduli spaces. From the point of view of an "operadchik", moduli spaces are even more interesting, because they form a very important and highly nontrivial operad. In fact, the moduli space $\overline{\mathcal{M}}_{g,n}$ of isomorphism classes of stable algebraic curves of genus g with n punctures (see a definition below) is not only an operad, but a *modular operad*. In fact, the moduli spaces $\overline{\mathcal{M}}_{g,n}$ became a model example for Getzler and Kapranov [**?**] to define the notion of a modular operad. We will mostly concentrate on the genus zero operad $\overline{\mathcal{M}}_{0,n}$ and, moreover, on a real version of it.

The moduli space $\mathcal{M}_{0,n}$ of compact complex algebraic curves of genus zero with n punctures is nothing but the space of configurations of n distinct labeled punctures on the complex projective lines $\mathbb{CP}^1$, considered up to the natural action of $\mathrm{PGL}_2(\mathbb{C})$. In other words,

$$\mathcal{M}_{0,n} = ((\mathbb{CP}^1)^n \setminus \Delta)/\,\mathrm{PGL}_2(\mathbb{C}),$$

where $\Delta = \bigcup_{i \neq j} \{(x_1, \dots, x_n) \in (\mathbb{CP}^1)^n \mid x_i = x_j\}$ is the large (or weak) diagonal.

The moduli space $\overline{\mathcal{M}}_{0,n}$ is the space of isomorphism classes of stable n-puinctured complex curves of genus zero. The punctures are labeled by numbers 1 through n, and a *stable* curve means that

1. (1) It is a curve which may have a finite number of singularities, which are all double points, these singularities may not occur at punctures;
2. (2) Each irreducible component is $\mathbb{CP}^1$ having at least 3 markings (counting punctures and double points);
3. (3) Consider the *dual graph* of a stable curve. This is a graph which has a vertex for each irreducible component and a half-edge going out of a vertex for each puncture and double point on the corresponding irreducible component, two half-edges being connected to form an edge if they correspond to a common double point, free half-edges are the legs of the graph, corresponding to the punctures. The dual graph of a stable curve of genus zero must be a tree.

The space $\overline{\mathcal{M}}_{0,n}$ is a complex complex compact manifold of dimension $n-3$, in which $\mathcal{M}_{0,n}$ is a Zariski-open subset. The space $\overline{\mathcal{M}}_{0,n}$ is the Deligne-Knudsen-Mumford compactification of $\mathcal{M}_{0,n}$. It may be also obtained as a quotient of the

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Fulton-MacPherson compactification by a free $\mathrm{PGL}_2(\mathbb{C})$ action — we will discuss this later.

A remarkable thing is that the moduli spaces $\mathcal{M}(n) := \overline{\mathcal{M}}_{0,n+1}$, $n \geq 2$, form an operad. To describe this structure, let us think of the first n punctures as inputs and the $n+1$st puncture as a unique output and call it ∞. Then the operad composition is given by *attaching*:

$$\gamma : \mathcal{M}(n) \times \mathcal{M}(m_1) \times \cdots \times \mathcal{M}(m_n) \to \mathcal{M}(m_1 + \cdots + m_n),$$

where you attach the outputs of the stable curves with $m_1, \dots, m_n$ inputs to the n inputs of the other stable curve, creating a genus zero stable curve with n new double points. The symmetric group acts on $\mathcal{M}(n)$ by permuting the labels at the inputs, leaving the output untouched.

As an illustration of the fact that this simple-minded structure gives rise to exciting things, let me quote the following theorem. Before describing it, let me note that if you have a topological operad, such as $\mathcal{M}$ above, you automatically get a graded operad (*i.e.*, one of graded vector spaces) by applying the homology functor with your favorited field coefficients. Thus, $\{H_\bullet(\mathcal{M}(n); k) \mid n \geq 2\}$ is a graded operad.

Theorem 1 (Getzler-Kontsevich-Manin)**.** *A graded vector space V has the structure of an algebra over the operad $H_\bullet(\mathcal{M}; k)$, if and only if V is a WDVV-algebra, see below.*

Definition 2. A *Witten-Dijkgraaf-Verlinde-Verlinde* (*WDVV*) *algebra* is a graded vector space V with a collection of graded symmetric operations $m_n(v_1, \dots, v_n)$ of degree $n-3$ for each $n \geq 2$, such that if you define the following formal bilinear product

$$w \cdot_v d := \sum_{n \geq 0} \frac{1}{n!} m_{n+2}(w, d, v, v, \dots, v),$$

then it will be associative in "Witten" w and "Dijkgraaf" $d \in V$ for each "Verlinde brother" $v \in V$.

The idea of the proof of this theorem is to assign the operations m_n to the fundamental class in $\mathcal{M}(n)$ and use Sean Keel's description of the homology ring of $\mathcal{M}$.