LECTURE 11: MODULI SPACES: ARBITRARY GENUS

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1. Moduli Spaces: Arbitrary Genus

For any pair of nonnegative integers g and n such that $2-2g-n<0$, the moduli space $\mathcal{M}_{g,n}$ of projective smooth algebraic curves of genus g with n (labeled) punctures is a space of dimension $3g-3+n$, whose complex points correspond bijectively to isomorphism classes of complex projective smooth algebraic curves of genus g with n labeled punctures. The term "space" here refers to a Deligne-Mumford stack (in algebraic geometry) or orbifold (in topology), the latter meaning that locally the space looks like a neighborhood of a point in the quotient of $\mathbb{R}^{6g-6+2n}$ by an lineraly free action of a finite group of linear transformations. Points whose neighborhoods look like the origin in that quotient are called orbifold singularities. Such points in the moduli space $\mathcal{M}_{g,n}$ correspond to algebraic curves having nontrivial automorphism groups. You can read more about the moduli space in J. Harris and I. Morrison's book [**?**]. Moduli spaces were created when the algebraic geometers studied all possible examples of algebraic varieties — they are just roots of polynomials, after all. Just kidding, although the phrase makes some sense...

The moduli space $\mathcal{M}_{g,n}$ of smooth curves is all, but compact, in general. Its compactification $\overline{\mathcal{M}}_{g,n}$ by Deligne and Mumford (and Knudsen, who generalized Deligne-Mumford's work to the punctured case) is a tour-de-force of geometric invariant theory. The space $\overline{\mathcal{M}}_{g,n}$ is a smooth projective Deligne-Mumford stack containing $\mathcal{M}_{g,n}$ as a Zariski-open subset. The complex points of $\overline{\mathcal{M}}_{g,n}$ may be described as isomorphism classes of stable complex smooth projective curves of genus g with n labeled punctures. A *stable* curve is one which may have a finite number of singular points, which are all double points, such that the normalizations of irreducible components of genus zero have at least three markings, which include punctures and double points, and the normalizations of irreducible components of genus one have at least one marking. This stability condition may be described as follows: if we remove all punctures and double points, then the Euler characteristic of each connected component of the complex curve must be negative.

Fact 1. (1) *The space $\overline{\mathcal{M}}_{g,n}$ admits a stratification by locally closed subspaces, which may be described as follows.*

$$\overline{\mathcal{M}}_{g,n} = \coprod_{n\text{-graphs } \Gamma} \mathcal{M}_\Gamma,$$

$$\mathcal{M}_\Gamma = \Big(\prod_{v\in\Gamma} \mathcal{M}_{g(v),n(v)}\Big)/\operatorname{Aut}\Gamma,$$

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where the strata $\mathcal{M}_\Gamma$ are labeled by connected graphs Γ with n labeled legs, decorated with integers $g(v) \geq 0$ at each vertex v, also satisfying the following conditions. If $n(v)$ denotes the valence of a vertex v, then $2 - 2g(v) - n(v) < 0$ for each vertex v and $b_1(\Gamma) + \sum_{v \in \Gamma} g(v) = g$.

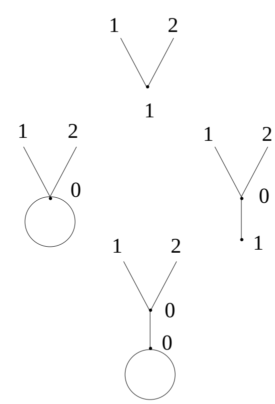
(2)
$$\partial \mathcal{M}_\Gamma = \bigcup_{\Gamma' : \Gamma = \Gamma'/e} \overline{\mathcal{M}}_{\Gamma'},$$
where Γ'/e is the contraction of an n-graph Γ' along an interior edge e; if this edge connects two distinct vertices of genus g_1 and g_2, then the vertex in Γ at which contraction occured gets a decoration of $g_1 + g_2$; if this edges is a loop at a vertex decorated by g_3, then the corresponding vertex in Γ gets decorated by $g_3 + 1$.

(3)
$$\dim \mathcal{M}_\Gamma = \sum_{v \in \Gamma} 3g(v) - 3 + n(v).$$

(4) *There is a unique open stratum. It corresponds to the graph with a unique vertex, decorated by g, and n legs emanating from it.*

(5) *The space $\mathcal{M}_\Gamma$ is the moduli space of stable curves whose* dual graph *(see the previous lecture, except that you decorate the vertex with the genus of the normalization of the corresponding irreducible component) is Γ.*

Example 2. The space $\overline{\mathcal{M}}_{1,2}$ is a space of (complex) dimension $3g - 3 + n = 2$. It has four strata, whose incidence (relative disposition) relation is described by the following diagram



On this diagram, we sketched the graphs corresponding to the strata, and each graph lies in the closure of every graph above it.

The collection of spaces $\coprod_{\substack{g \geq 0 \\ 2-2g-n<0}} \overline{\mathcal{M}}_{g,n+1}$ for $n \geq 0$ forms an operad with respect to the attaching operations

$$\circ_i : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$

$i = 1, \dots, n_1$. However, a certain nonsymmetry (not to say, ugliness) of the definition gets corrected when you define on the moduli spaces either the structure of a PROP or a modular operad, see Getzler-Kapranov [**?**].

There is a much smaller suboperad of the moduli space operad comprised by the stable curves of genus zero. The above fact gets translated in this case into the following one.

Fact 3. (1) *The space* $\overline{\mathcal{M}}_{0,n}$ *admits a stratification by locally closed subspaces, which may be described as follows.*

$$\overline{\mathcal{M}}_{0,n} = \coprod_{n\text{-trees } T} \mathcal{M}_T,$$

$$\mathcal{M}_T = \prod_{v \in T} \mathcal{M}_{0,n(v)},$$

where the strata $\mathcal{M}_T$ *are labeled by connected rooted trees* T *with* n *labeled leaves, such that the valence* $n(v)$ *of each vertex* v *is at least three.*

(2)

$$\partial \mathcal{M}_T = \bigcup_{T' : T = T'/e} \overline{\mathcal{M}}_{T'},$$

where T'/e *is the contraction of an* n*-tree* T' *along an interior edge* e*.*

(3)

$$\dim \mathcal{M}_T = \sum_{v \in T} (n(v) - 3).$$

(4) *There is a unique open stratum. It corresponds to the* n*-corolla, the tree with a unique vertex, a root, and* n *leaves.*

(5) *The space* $\mathcal{M}_T$ *is the moduli space of stable curves of genus zero whose* dual graph *is* T*.*