LECTURE 12: A REAL COMPACTIFICATION OF MODULI SPACES

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1. A real compactification of moduli spaces

There must be a real version of the above compactification of moduli space $\mathcal{M}_{g,n}$, in which the complex points will be the isomorphism classes of stable algebraic curves of genus g with n labeled punctures with a pair of real tangent directions fixed at each double point, one to each piece of the curve intersecting at the double point, up to the diagonal action of $U(1)$, by rotating the tangent direction by θ on one piece and $-\theta$ on the other. This must be a real manifold with corners, one glued out of local charts, each of which diffeomorphic to a neighborhood of a point in a closed cube. I believe there is such a compactification, because of the real version of the Fulton-MacPherson compactification, which conveniently provides this space for $g = 0$ and will be studied later.

Problem 1. Define this compactification as a Deligne-Mumford stack. [Hint: This problem should not be hard, but its solution is not known, at least to my knowledge. Perhaps, this compactification is just the real blowup along the boundary divisor of the Deligne-Mumford compactification.]

These compactifications would not form an operad or a PROP. The problem is that when you attach the complex curves at punctures, there will be no natural choice of the real tangent directions. However, the moduli spaces of stable curves with real tangent directions at the punctures and pairs of real tangent directions at the double points, considered up to the diagonal actions of S^1, must form a PROP and an operad. These are not compactifications of the moduli space $\mathcal{M}_{g,n}$, but rather of a principal $(S^1)^n$-bundle over it.

In genus zero, there is a third collection of moduli spaces, forming an operad, which will be of most interest to us, because of a direct connection with the little disks operad and the L_∞ operad. Let $\mathcal{M}(n)$, $n \geq 2$, be the moduli space of stable algebraic curves of genus zero with $n+1$ labeled punctures and the choice of a real tangent direction at the $n+1$st puncture denoted ∞, as well as a real tangent direction at each double point to the component which is further away from the ∞ puncture. This space is indeed obtained via a sequence of blowups of the boundary divisor of the Deligne-Mumford compactification of $\mathcal{M}_{,0}n+1$, which is further extended to a principal S^1-bundles of real tangent directions at the $n+1$st puncture, see Kimura-Stasheff-A.V. [?]. The *topological stratification* of the space $\mathcal{M}(n)$ is obtained by fixing the dual graph of the curve, which is, again a n-tree T, producing a stratum $\mathcal{M}_T^r = \prod_{v \in T} \mathcal{M}_{0,n(v)}^r$, each $\mathcal{M}_{0,n(v)}^r$ being a principal S^1-bundle corresponding to the choice of a real tangent direction at ∞ over the moduli

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space $\mathcal{M}_{0,n(v)}$. The *topological filtration* is given by

$$F^p = \{\text{the closure of the union } \coprod_T \mathcal{M}_T^r \text{ of strata of dimension } p\}.$$

This filtration is compatible with the operad structure in the way that it is invariant under the symmetric group actions and satisfies the condition

$$F^p(m) \circ_i F^q(n) \subset F^{p+q}(m+n-1),$$

where $F^p(m) \subset \mathcal{M}(m)$ and $i = 1, \dots, m$. In this situation, applying the functor of spectral sequences of a filtered space, we get a funny creature, which may be called an operad of spectral sequences, see [**?**]. The first term of this spectral sequence is a direct sum of operads of complexes, one of them being the L_∞ operad, according to the following result, in which we assume coefficients in a ground field k of characteristic $\neq 2$.

Proposition 1 (Kimura-Stasheff-A.V. [**?**])**.** *The spectral sequence $E^r_{p,q}$, $n-1$ lep $\leq 2n-3$, $-p \leq q \leq 0$, associated with the above filtration has the following properties.*

(1) $E^r_{\bullet,\bullet} \Rightarrow H_\bullet(\mathcal{M}(n)) = H_\bullet(\mathcal{M}_{0,n+1}) = H_\bullet(D(n))$, *where D is the little disks operad.*

(2) $E^1_{p,q} = H_{p+q}(F^p, F^{p-1})$ *and its $q = 0$ row*

$$0 \to E^1_{2n-3,0} \to \cdots \to E^1_{p,0} \to E^1_{p-1,0} \to \cdots \to E^1_{n-1,0} \to 0$$

is naturally isomorphic to the nth component $L_\infty(n)[n-1]$ of the L_∞ operad shifted to the left by $n-1$. These isomorphisms for different n establish an isomorphism of operads.

(3) *The spectral sequence E^r degenerates at E^2: $E^2 = E^\infty$.*

(4) *The homology of E^1 is concentrated at the right end and is isomorphic to $H_\bullet(\mathcal{M}_{0,n+1}) = H_\bullet(D(n))$.*

We have not defined the L_∞ *operad* in this course yet. It is defined similar to the A_∞ operad, except that we consider abstract (*i.e.*, nonplanar) trees with an orientation on the set of their edges, as in the lecture on the Lie bialgebra PROP. The differential for the L_∞ operad is defined by the same vertex expansion operator, as in the lecture on the A_∞ operad. Note that the vertex expansion operation produces a different result when performed over abstract trees.

Now the connection with the L_∞ operad can be made by the following argument. By definition of the spectral sequence,

$$\begin{aligned} E^1_{p,0} &= H_p(F^p, F^{p-1}) = H^0(F^p \setminus F^{p-1}) \\ &= H^0\Big(\coprod_{\substack{\text{oriented } n\text{-trees } T \text{ with} \\ v(T) = 2n-2-p \text{ vertices}}} \mathcal{M}_T^r\Big) = \bigoplus_{\substack{\text{oriented } n\text{-trees } T \text{ with} \\ v(T) = 2n-2-p \text{ vertices}}} k, \end{aligned}$$

where k is the ground field. At least as a vector space, this is exactly the term of degree $2n-2-p+1-n = n-1-p$ of $L_\infty(n)$, as described by oriented trees. Observe that the operad composition in $\mathcal{M}$ corresponds to grafting of the trees, while the symmetric group action on $\mathcal{M}(n)$ corresponds to relabeling the leaves. It is not hard to check that with appropriate choice of orientations on the strata, the differential on E^1 will be equal to that defined for the L_∞ operad by the vertex expansion operator.