LECTURE 15: TCFT'S OR STRING BACKGROUNDS

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1. TCFT's or String Backgrounds

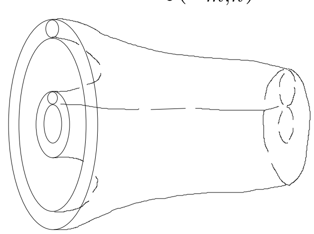
We will now consider TCFT's or string backgrounds, or in other words, CFT's with ghosts and total central charge $c = 0$. Consider the Segal PROP $\mathcal{P} = \{\mathcal{P}_{m,n} \mid m, n \geq 0\}$, algebras over which defined CFT's. Take the corresponding DG PROP of singular chains $C_\bullet(\mathcal{P})$. The coefficients will always be complex when we talk about TCFT's. We will give three definitions of a TCFT, gradually closer to what the physicists work with. These definitions will not be equivalent - after all, this is physics! However, you can do many things equally well with all of them. The first one is really short.

1.1. The first definition of a TCFT.

Definition 1. A *TCFT* is an algebra V over the PROP $C_\bullet(\mathcal{P})$.

Thus, not only points of the PROP $\mathcal{P}$ give rise to operators between tensor powers of the vector space V, as it was in CFT, but also all singular chains on $\mathcal{P}$ do:

$$C_\bullet(\mathcal{P}_{m,n}) \quad \to \quad \mathrm{Hom}(V^{\otimes m}, V^{\otimes n}),$$



$$\longmapsto \quad |C\rangle : V^{\otimes m} \to V^{\otimes n}.$$

Chains C in $\mathcal{P}_{m,n}$ Linear operators $|C\rangle$

On the figure, the surface is nothing but a pair of pants (so $m = 2$, $n = 1$) and the chain is just a circle. The pants moving along the circle in the moduli space sweep out a "surface of revolution", which I attempted to sketch above.

Since we are now in the category of complexes, V must be a complex with a differential Q, $\deg Q = 1$, $Q^2 = 0$, and the structure maps of a PROP algebra should be compatible with not only the PROP structure, but also the degrees and the differentials, Q on the the endomorphism PROP of V and ∂ on the PROP of chains of $\mathcal{P}$. The complex V is called a *state space*, or a *BRST complex*, Q a *BRST differential*, and the cohomology $H^\bullet(V, Q)$ the *physcial state space*, or the *BRST cohomology*. The DG structure compatibility conditions will then read as follows.

$$\begin{aligned} \deg |C\rangle &= -\deg C, \\ Q|C\rangle &= |\partial C\rangle, \end{aligned}$$

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where the negative sign is just due to the fact that we have chosen the homological notation for singular chains and the cohomological one for the BRST complex.

1.2. **The second definition of a TCFT.** If I were you, I would find it hard to believe that the physicists may come up with something as simple in its abstractness as the above definition. Indeed, the physicists try to put in as much data as they can possibly use and, in general, avoid definitions, nothing to say concise ones. I will present another two definitions of a TCFT, which will be gradually more "physical".

Definition 2. A *TCFT* is a complex (V, Q) of vector spaces, $\deg Q = 1$, $Q^2 = 0$, along with a collection of smooth differential forms $\Omega_{m,n} = \sum_{r \geq 0} \Omega^r_{m,n}$, $\Omega^r_{m,n} \in \Omega^r(\mathcal{P}_{m,n}) \otimes \operatorname{Hom}^{-r}(V^{\otimes m}, V^{\otimes n})$, for each pair $m, n \geq 0$, where Hom^{-r} denotes the space of linear maps of degree $-r$. These data must satisfy the following axioms.

(1)
$$d\Omega_{m,n} = Q\Omega_{m,n},$$
which is understood as a family of identities $d\Omega^r_{m,n} = Q\Omega^{r+1}_{m,n}$, where d is the de Rham differential, acting geometrically, and Q is the BRST differential, acting algebraically on the coefficients.

(2)
$$\sigma^*\Omega_{m,n} = \sigma\Omega_{m,n},$$
where $\sigma \in S_m$ or S_n, acting geometrically on the left-hand side by relabeling the holes on the Riemann surface and algebraically on the right-hand side by permuting the factors in the corresponding tensor power of V.

(3)
$$\gamma^*\Omega_{m,k} = \gamma(\Omega_{m,n} \boxtimes \Omega_{n,k}),$$
where γ denotes the PROP composition and is again the geometric one, $\gamma : \mathcal{P}_{m,n} \times \mathcal{P}_{n,k} \to \mathcal{P}_{m,k}$, on the left-hand side and the algebraic one, $\gamma : \operatorname{Hom}(V^m, V^n) \otimes \operatorname{Hom}(V^n, V^k) \to \operatorname{Hom}(V^m, V^k)$, on the right-hand side.

Remark 3. Let us assume once and for all that the singular chains in Definition 1 are smooth (*i.e.*, piecewise smooth). This will enable to integrate differential forms over them and construct a TCFT in the sense of Definition 1 from Definition 2, using the formula:
$$C \mapsto \int_C \Omega$$
to produce a morphism $C_\bullet(\mathcal{P}) \to \mathcal{E}nd_V$ of PROP's.

1.3. **The third definition of a TCFT.** Before giving this definition, we need to introduce some more characters into the play. Let V be the Lie algebra of vector fields on the circle, which we will call the Virasoro algebra, slightly abusing standard terminology.

Definition 4. A *TCFT* is a CFT based on a state space $\mathcal{H}$ with the following extra *data*:

(1) The structure of a complex on $\mathcal{H}$, *i.e.*, a $\mathbb{Z}$-grading $\mathcal{H} = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i$ and a BRST differential $Q : \mathcal{H} \to \mathcal{H}$, $Q^2 = 0$, of degree 1.

(2) An action of the Clifford algebra $C(V \oplus V^*)$, which is denoted usually by $b : V \otimes \mathcal{H} \to \mathcal{H}$ and $c : V^* \otimes \mathcal{H} \to \mathcal{H}$ for generators of the Clifford algebra, the degree of b is -1, and the degree of c is 1.
(3) An action of the Lie algebra V on $\mathcal{H}$: $T : V \otimes \mathcal{H} \to \mathcal{H}$.

The graded space $\mathcal{H}$ with the operator Q is called a *BRST complex*. For $\psi \in \mathcal{H}_i$, the degree $\operatorname{gh} \psi := i$ is called the *ghost number*.

Remark 5. Usually, the space $\mathcal{H}$ of a string background is constructed from a CFT based on a space of "matter" $\mathcal{H}_m$ and a "ghost" CFT based on a space $\mathcal{H}_{\mathrm{gh}}$ as the tensor product $\mathcal{H} = \mathcal{H}_m \otimes \mathcal{H}_{\mathrm{gh}}$, the grading coming from the second factor. In that case, the CFT's $\mathcal{H}_m$ and $\mathcal{H}_{\mathrm{gh}}$ must be more general than the ones we consider, because $\mathcal{H}_m$ and $\mathcal{H}_{\mathrm{gh}}$ have nontrivial central charges, i.e., they are rather representations of the central extension of the Lie algebra V; this central extension is what is usually called the Virasoro algebra. But for the resulting string background, the central charge can be made 0 by an appropriate choice of $\mathcal{H}_m$.

These data must satisfy the following *axioms*:

(4) $[T(v_1), b(v_2)] = b([v_1, v_2])$ and $[T(v_1), c(v_2^*)] = c((\operatorname{ad}^* v_1) v_2^*)$, as in any BRST complex over V or Vir.
(5) $[Q, T(v)] = 0$, $\{Q, b(v)\} = T(v)$, $\{Q, c(v^*)\} = c(dv^*)$, where $dv^* \in \Lambda^2(V)^*$ is the Lie algebra differential of the 1-cochain $v^* \in V^*$, as in any BRST complex.
(6) $\operatorname{gh} |\Sigma\rangle = 0$.
(7) $\mathbf{b}(\mathbf{v})|\Sigma\rangle = \mathbf{b}(\overline{\mathbf{v}})|\Sigma\rangle = 0$ for any $\mathbf{v} \in V^{n+1}$ extended holomorphically on Σ.
(8) $Q|\Sigma\rangle = 0$.

Remark 6. Following A. S. Schwarz's idea, one can define a more general string background in the same fashion as we defined a CFT in Remark **??**: a generalized string background is an algebra over the operad $\mathcal{P}^{sr}_{n+1}$ of semirigid $N = 2$ super Riemann surfaces introduced by Distler and Nelson [DN91]. The operators $b(v)$ and Q on states $|\Sigma\rangle$ can be read off as infinitesimal transformations of the super structure on a semirigid surface Σ, similar to the action $T(v)$ of the Virasoro in the usual CFT case. The connection to semirigid supergravity of Distler and Nelson has been pointed out in Horava's paper [**?**].

1.4. **A morphism of complexes.** One of the nicest implications of a string background is the construction of a morphism of complexes $\mathcal{H}^{\otimes n} \to \Omega^\bullet(\mathcal{P}_{n+1})$, $n \geq 1$, from the tensor power of the BRST complex $\mathcal{H}$ to the de Rham complex of the space $\mathcal{P}_{n+1}$. We will use a somewhat partially dual picture and construct for each n a $\operatorname{Hom}(\mathcal{H}^n, \mathcal{H})$-valued form $\Omega_{n+1} = \sum_{r \geq 0} \Omega^r_{n+1}$, $\deg \Omega^r_{n+1} = r$, on the space $\mathcal{P}_{n+1}$:

$$\Omega_{n+1}(\mathbf{v}_1, \dots, \mathbf{v}_r) := \Omega^r_{n+1}(\mathbf{v}_1, \dots, \mathbf{v}_r) := \mathbf{b}(\widetilde{\mathbf{v}}_1) \dots \mathbf{b}(\widetilde{\mathbf{v}}_r)|\Sigma\rangle, \tag{1.1}$$

where $\mathbf{v}_i$, $1 \leq i \leq r$, is a tangent vector to the space $\mathcal{P}_{n+1}$ at the point Σ and $\widetilde{\mathbf{v}}_i$ is its pullback to an element of V^{n+1}, acting by infinitesimal reparameterizations at punctures.

Theorem 7.

$$(d - Q)\Omega_{n+1} = 0$$

Remark 8. The only structure we needed from a string background for our purposes was this collection of forms Ω_{n+1}. One can show that a collection of such forms together with a natural operad condition is equivalent to a generalized string background, see Segal [Seg93] or Getzler [Get94].

Proof. More precisely, fixing n and omitting the subscript $n+1$, we have to prove that for each $r \geq 0$

$$d\Omega^{r-1} = Q\Omega^r, \tag{1.2}$$

where $\Omega^{-1} = 0$ by definition. Let us use induction on r. For $r = 0$, (1.2) is $0 = Q|\Sigma\rangle$, which has been postulated. Suppose that Equation (1.2) is true for some $r \geq 0$. To make the induction step, it suffices to prove that for any tangent vector $\mathbf{v}$ to $\mathcal{P}_{n+1}$,

$$\iota(\mathbf{v}) d\Omega^r = \iota(\mathbf{v}) Q\Omega^{r+1},$$

where ι is the contraction of a differential form with a tangent vector. From (1.1), it is clear that for all $r \geq 0$

$$\iota(\mathbf{v})\Omega^r = \mathbf{b}(\widetilde{\mathbf{v}})\Omega^{r-1}. \tag{1.3}$$

According to Axiom (**??**) of Section **??**, we have that the Lie derivative $\mathcal{L}(\mathbf{v}) := \{d, \iota(\mathbf{v})\}$ of a form along a tangent vector $\mathbf{v}$ is equal to $\mathbf{T}(\widetilde{\mathbf{v}})$:

$$\{d, \iota(\mathbf{v})\} = \mathcal{L}(\mathbf{v}) = \mathbf{T}(\widetilde{\mathbf{v}}) = \{Q, \mathbf{b}(\widetilde{\mathbf{v}})\}. \tag{1.4}$$

Since the operator d acts geometrically and $\mathbf{b}$ acts in coefficients, we have

$$[d, \mathbf{b}(\widetilde{\mathbf{v}})] = 0.$$

Similarly,

$$[Q, \iota(\mathbf{v})] = 0.$$

Using these equations, we obtain

$$\begin{aligned}
\iota(\mathbf{v}) d\Omega^r &= \mathcal{L}(\mathbf{v})\Omega^r - d\iota(\mathbf{v})\Omega^r \\
&= \mathbf{T}(\widetilde{\mathbf{v}})\Omega^r - d\mathbf{b}(\widetilde{\mathbf{v}})\Omega^{r-1} \\
&= \mathbf{T}(\widetilde{\mathbf{v}})\Omega^r - \mathbf{b}(\widetilde{\mathbf{v}}) d\Omega^{r-1} \\
&= \mathbf{T}(\widetilde{\mathbf{v}})\Omega^r - \mathbf{b}(\widetilde{\mathbf{v}}) Q\Omega^r \\
&= Q\iota(\mathbf{v})\Omega^{r+1} \\
&= \iota(\mathbf{v}) Q\Omega^{r+1}. \quad \square
\end{aligned}$$

Some crucial properties of these differential forms are their equivariance under the operad map and the actions of the permutation group.

Theorem 9.

$$\gamma^*\Omega_{n+1} = \gamma(\Omega_{k+1}; \Omega_{n_1+1}, \dots, \Omega_{n_k+1})$$

where $n = n_1 + \dots + n_k$, $\gamma^* : \Omega^\bullet(\mathcal{P}_{n+1}) \to \Omega^\bullet(\mathcal{P}_{k+1} \times \mathcal{P}_{n_1+1} \times \cdots \mathcal{P}_{n_k+1})$ *is the pullback of the operad map* $\mathcal{P}_{k+1} \times \mathcal{P}_{n_1+1} \times \dots \times \mathcal{P}_{n_{k+1}} \to \mathcal{P}_n$ *and the other* γ *is the operad map in the endomorphism operad. Similarly, for all* σ *in* Σ_n*, the permutation group, we have*

$$\sigma^*\Omega_{n+1} = \Omega_{n+1} \circ \sigma.$$

Proof. Both are proved by using the axioms of a string background. $\square$

Remark 10. Theorems 7 and 9 insure that the maps $C_\bullet(\mathcal{P}_{n+1}) \to \mathrm{Hom}(\mathcal{H}^n, \mathcal{H})$ given by $C \mapsto \int_C \Omega_{n+1}$ makes $\mathcal{H}$ into an algebra over $\{C_\bullet(\mathcal{P}_{n+1})\}$ which, in turn, makes absolute BRST cohomology, $H^\bullet$, into an algebra over the operad $\{H_\bullet(\mathcal{P}_{n+1})\}$.

Remark 11. Dually, the above properties of the forms Ω_{n+1} along with their behavior with respect to the identity of the operad $\{\mathcal{P}_{n+1}\}$ can be formulated as follows. If the Ω_{n+1}'s were regarded as maps from $\mathrm{Hom}(\mathcal{H}^n, \mathcal{H})^* \to \Omega^\bullet(\mathcal{P}_{n+1})$, then they would define the structure of a coalgebra over the DG cooperad of differential forms on $\mathcal{P}_{n+1}$ on the complex $\mathcal{H}$.

References

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