

True/False

Mark each statement as True or False.

#7. The weighted voting system [6: 3, 2, 1] is equivalent to [3: 1, 1, 1]. True False

TRUE. In each system, the players share the power equally.

#8. The grand coalition consists of all of the players in a weighted voting system. True False

TRUE. This is the definition of the grand coalition.

#9. A player with veto power always has a Banzhaf power index of 100%. True False

FALSE. A *dictator* has 100% of the power.

#10. If two players in a weighted voting system have the same number of votes, then they have the same Shapley-Shubik power index. True False

TRUE. Two players with the same number of votes have the same power.

#11. The power distribution in a weighted voting system that is found using the Banzhaf power index is always the same as the distribution that is found using the Shapley-Shubik power index. True False

FALSE. In some weighted voting systems, the distributions are different.

#12. In the United Nations Security Council, the power is divided evenly among all 15 members. True False

FALSE. The five permanent members control most of the power.

Part II

Problems 13 and 14 deal with the weighted voting system [12: 7, 6, 5].

#13 (5 pts)

Find the Banzhaf power index for each player. Be sure to show all of your work.

Here are the winning coalitions with the critical players underlined:

$$\{\underline{P_1}, \underline{P_2}\} \quad \{\underline{P_1}, \underline{P_3}\} \quad \{\underline{P_1}, P_2, P_3\}$$

There are a total of 5 critical players – this is the denominator. The numerator is the number of times a player is critical. So the Banzhaf power index for each player is

$$P_1 : \frac{3}{5} = 60\% \quad P_2 : \frac{1}{5} = 20\% \quad P_3 : \frac{1}{5} = 20\%$$

#14 (5 pts)

Find the Shapley-Shubik power index for each player. Be sure to show all of your work.

Here are the sequential coalitions with the pivotal players underlined:

$$\langle P_1, \underline{P_2}, P_3 \rangle \quad \langle P_1, \underline{P_3}, P_2 \rangle \quad \langle P_2, \underline{P_1}, P_3 \rangle \quad \langle P_2, P_3, \underline{P_1} \rangle \quad \langle P_3, \underline{P_1}, P_2 \rangle \quad \langle P_3, P_2, \underline{P_1} \rangle$$

There are $3! = 6$ sequential coalitions – this is the denominator. The numerator is the number of times a player is pivotal. So the Shapley-Shubik power index of each player is

$$P_1 : \frac{4}{6} \approx 66.7\% \quad P_2 : \frac{1}{6} \approx 16.7\% \quad P_3 : \frac{1}{6} \approx 16.7\%$$

#15 (3 pts) What is a dictator? Which, if any, players in the weighted voting system [8: 9, 4, 2] are dictators? Explain your answer.

A dictator has all of the power in a weighted voting system.

In this system, P_1 is a dictator because he has more votes than the quota so he can pass or fail a motion on his own.

#16 (3 pts) What is a dummy? Which, if any, players in the weighted voting system [4: 3, 3, 1] are dummies? Explain your answer.

A dummy has no power in a weighted voting system.

In this system, there is no dummy. Each player has the potential to effect the outcome of the vote. One way to see this is to notice that all players are critical in any coalition that consists of them and one other player. If a player is ever critical, they can't be a dummy.

#17 (3 pts) What does it mean for a player to have veto power? Which, if any, players in the weighted voting system [7: 5, 4, 2] have veto power? Explain your answer.

A player has veto power if they have to vote yes in order for the motion to pass.

In this system, P_1 has veto power because the other players don't have enough votes to meet the quota without him.

#18 (3 pts)

The Mudville City Council has 1 mayor plus 9 regular councilmembers. The mayor does not usually vote. In order to pass a motion, 5 or more votes from the regular councilmembers are needed. However, the mayor can veto the motion. In order to override the veto, 7 regular councilmembers must vote for the motion.

Find the Shapley-Shubik power index of the mayor. It might be helpful to think of the situation this way: to pass, a motion requires the support of either 5 or 6 regular councilmembers plus the mayor, or else 7 or more councilmembers (with or without the mayor).

The mayor is pivotal when there are enough votes from regular councilmembers to pass a motion, but not to override a veto. That is, the mayor is pivotal in the 6th and 7th positions out of 10.

If the mayor is in the 6th position, there are 9! ways to arrange the regular council members around the mayor. Similarly, if the mayor is in the 7th position, there are 9! ways to arrange the regular council members around the mayor. This means the mayor is pivotal in $2 \times 9!$ of the $10!$ sequential coalitions that include the mayor as well as the regular council members.

So the mayor's Shapley-Shubik power index is

$$\frac{2 \times 9!}{10!} = \frac{2}{10} = \frac{1}{5} = 20\%$$

#19 (4 pts)

The Hobbits of the Shire have decided to institute a weighted voting system. The Shire consists of four districts: the Northfarthing (with 41.9% of the population), the Southfarthing (with 25.8%), the Eastfarthing (with 23.4%), and the Westfarthing (with 8.9%). Two possible weighted voting systems have been proposed: [51: 42, 26, 23, 9] and [51: 48, 47, 3, 2]. With the help of a passing wizard, the Hobbits have calculated the percentage of votes each district would have in each system and the Banzhaf power distributions of the two systems. This information is given in the tables below.

[51: 42, 26, 23, 9]

| District | Northfarthing | Southfarthing | Eastfarthing | Westfarthing |
|----------------------------|---------------|---------------|--------------|--------------|
| % Population | 41.9 | 25.8 | 23.4 | 8.9 |
| % of Votes | 42 | 26 | 23 | 9 |
| Banzhaf power index (in %) | 50 | 16.7 | 16.7 | 16.7 |

[51: 48, 47, 3, 2]

| District | Northfarthing | Southfarthing | Eastfarthing | Westfarthing |
|----------------------------|---------------|---------------|--------------|--------------|
| % Population | 41.9 | 25.8 | 23.4 | 8.9 |
| % of Votes | 48 | 47 | 3 | 2 |
| Banzhaf power index (in %) | 41.7 | 25 | 25 | 8.3 |

Which of the two weighted voting systems should the Shire adopt and why? Justify your answer based on information from the tables.

The Hobbits should adopt the second weighted voting system, [51: 48, 47, 3, 2]. The Banzhaf power index for each district in this system is very close to the % population for the district. Thus the power is distributed fairly throughout the Shire.

In the first weighted voting system, although the % of votes is close to the % population, the power distribution is not. Having the power distributed fairly is much more important than having the number of votes distributed fairly.