09fall, Final Exam Answer

1 I. Definitions: Complete the following sentences.

a:
Let $b \in \bar{R}, S \subseteq \bar{R}$. We say that b is the supremum of S if $\forall$ upper bound a for S, we have $a \geq b$.
b:
Let V and W be subspaces of Euclidean spaces. A map $L : V \to W$ is a linear transformation if both of the following hold: (1) for all $v, v^{'} \in V$, $L(v + v^{'}) = L(v) + L(v^{'})$, (2)for all scalars c, for all $v \in V$, $L(cv) = c[L(v)]$.
c:
An oriented parallelpiped in R^3 is an ordered triples of ordered triples of scalars.
d:
A matrix $M \in R^{n\times n}$ is a rotation matrix if it is an orthogonal matrix of determinant 1.
e:
Let M be a square matrix and let a be an eigenvalue of M. Then the a-eigenspace of M is $ker(M - aI)$.
f:
A function $B : R^m \times R^n \to R$ is bilinear if both of the following two conditions hold: (1) for all $v \in R^m$, the function $B(v, \bullet) : R^n \to R$ is linear, (2) for all $w \in R^n$, the function $B(\bullet, w) : R^m \to R$ is linear.
g:
The characteristic polynomial of a matrix M is the polynomial $\rho_M(t)$defined by $\rho_M(t) = det(tI - M)$.
h:
Let X and Y be PCRVs. The covariance of X and Y is $Cov[X, Y] = \frac{1}{2}(Var[X + Y] - Var[X] - Var[Y])$.

2 II. True or False.

a: True.
b: True.

c: True.
d: True.
e: False.
f: True.
g: False.
h: False.

3 III. Computations.

1:
$\begin{pmatrix} 5+7 \\ 5 \end{pmatrix} - \begin{pmatrix} 4+7 \\ 4 \end{pmatrix} = \frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1} - \frac{11\times10\times9\times8}{4\times3\times2\times1} = \frac{7\times11\times10\times9\times8}{5\times4\times3\times2\times1}$

2:
a:
$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 7 & 4 \\ 1 & 1 & 6 & 4 \end{pmatrix} - (R_2 - 2\times R_1, R_3 - R_1) \rightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix} - (1/4 \times R_3) \rightarrow$
$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3/4 \end{pmatrix} - (R_1 - 2\times R_3, R_2 - 3R_3) \rightarrow \begin{pmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{pmatrix} -$
$(R_1 - R_2) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{pmatrix} - (C_4 - 1/4C_1, C_4 + 1/4C_2, C_4 - 3/4C_3) \rightarrow$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$
b:
dim Im X=3.
c:
dim Ker X=1.

3:
$N = MM^t = \begin{pmatrix} 1 & 3 & 1 \\ & 1 & 3 \end{pmatrix}.$
First, find rotation K, such that $KNK^t = diagonal$.
Solve $det(N - \rho I) = 0$, we have eigenvalues $\rho_1 = 2, \rho_2 = 4$.
For $\rho_1 = 2$, the eigenvector $v_1 = (1/\sqrt{2}, -1/\sqrt{2})$, For $\rho_2 = 4$, the eigenvector $v_2 = (1/\sqrt{2}, 1/\sqrt{2})$, So $K = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.
Then, $KM(KM)^t = diagonal$ means the rows of KM are orthogonal to one another.

4:

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X = A + B$.

$e^A = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix}$, $e^B = I + B + \frac{B^2}{2!} + ... = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Note that $AB = BA$. So, $e^X = e^{A+B} = e^A e^B = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & e^2 \\ 0 & e^2 \end{pmatrix}$.

5:

a:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 9 \end{pmatrix}.$$

b:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 4 & 6 \\ 4 & 5 & 6 & 8 & 10 & 12 \\ 7 & 8 & 9 & 14 & 16 & 18 \\ -1 & -2 & -3 & -2 & -4 & -6 \\ -4 & -5 & -6 & -8 & -10 & -12 \\ -7 & -8 & -9 & -14 & -16 & -18 \end{pmatrix}.$$

6:

$v_1 = (\sqrt{1/3}, 0, \sqrt{2/3}), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$.

$w_1 = \frac{v_1}{|v_1|} = v_1 = (\sqrt{1/3}, 0, \sqrt{2/3})$;

$v_2 = e_2 - (e_2, w_1)w_1 = e_2$;

$w_2 = \frac{v_2}{|v_2|} = v_2 = e_2 = (0, 1, 0)$;

$v_3 = e_3 - (e_3, w_1)w_1 - (e_3, w_2)w_2 = (-\sqrt{2}/3, 0, 1/3)$;

$w_3 = \frac{v_3}{|v_3|} = (-\sqrt{2/3}, 0, 1/\sqrt{3})$.

w_i are the orthonormal basis of R^3.

7:

a:

det(M)=-1.

b:

The inverse of M is (-1)×the transpose of cofactor matrix of M, which is $\begin{pmatrix} -3 & 2 & 8 \\ 2 & -1 & -6 \\ -2 & 1 & 7 \end{pmatrix}$.

8:

a:

$det(M - \rho I) = \rho^2 - 5\rho + 6 = 0$, two eigenvalues of M are $\rho_1 = 2, \rho_2 = 3$.

b:

For $\rho_1 = 2$, let $v = (v_1, v_2)$ be the eigenvector. then, $\begin{pmatrix} 1-2 & 2 \\ -1 & 4-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$

0, we have $v = (2,1)$. So, 2-eigenspace of M is span{(2,1)}.

9:
$X_1, ..., X_{100}$ are standard, $\rightarrow Var[X_i] = 1$.
$X_1, ..., X_{100}$ are independent $\rightarrow Var[X_1 + ... + X_{100}] = Var[X_1] + ... + Var[X_{100}] = 100$.
So, $SD[X_1 + ... + X_{100}] = \sqrt{Var[X_1 + ... + X_{100}]} = \sqrt{100} = 10$.

10:
Let $Z = Y - X$.
$Cov[X, X] = 1/2 \times (Var[2X] - Var[X] - Var[X]) = Var[X] = 1$.
X,Z are independent $\rightarrow Cov[X, Z] = 0$.
So, by linearity of Cov:
$Cov[X, Y] = Cov[X, X + Z] = Cov[X, X] + Cov[X, Z] = 1$.