

Midterm 2

1 I. Definitions: Complete the following sentences.

- a: Let $A, C \in R^{n \times n}$. The **left conjugate** of A by C is CAC^{-1} .
- b: Let $N \in R^{n \times n}$. We say that N is **nilpotent** if some power of it is zero.
- c: The **dimension** of a subspace V of $R^{n \times n}$ is the size of any basis of V .
- d: The **standard oriented parallelogram** in R^2 is $((1,0), (0,1))$.
- (e): For any $M \in R^{n \times n}$, $e^M := \lim_{n \rightarrow \infty} [I + (M/n)]^n = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots$

2 II. True or False.

- a: False
- b: False
- c: True
- d: False
- e: True

3 III. Computations.

1.

a. $P \oplus Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -3 & -2 \end{pmatrix}$

b. $P \otimes Q = \begin{pmatrix} 6 & 4 & 0 & 0 \\ -6 & -4 & 0 & 0 \\ 0 & 0 & -6 & -4 \\ 0 & 0 & 6 & 4 \end{pmatrix}$

2.

a. $I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

b. $M^2/(2!) = \begin{pmatrix} 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, M^3/(3!) = \begin{pmatrix} 0 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$

$M^4/(4!) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1/24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$

c. $M^5 = 0.$

d. For $M^n = 0, \forall n \geq 5, e^M = I + M + M^2/2! + M^3/3! + M^4/4! = \begin{pmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 \\ 0 & 1 & 1 & 1/2 & 1/6 \\ 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

3. Find the fully canonical form by row and column operations: $\begin{pmatrix} 1 & 2 & 4 & 2 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 2 & 3 & 6 & 4 & 0 \\ 3 & 4 & 8 & 6 & 0 \end{pmatrix} \longleftrightarrow$

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$ so, $\dim \text{Im} = 2, \dim \text{Ker} = 3.$

4. There are three types of row operations and three types of column operations. **Only multiplying one row or column by a constant will change the determinant.** As in question 3, no such operations are used.

So $\det \begin{pmatrix} 1 & 2 & 4 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 6 & 4 \\ 3 & 4 & 8 & 6 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$

Also, one can expand the matrix by the first row or column to obtain the determinant.

5.

a. Matrix of minors: $\begin{pmatrix} d & c \\ b & a \end{pmatrix}.$

b. Matrix of cofactor: $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$

c. $C^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, C^t M = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}.$

6. The signed volume is equal to $\det \begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & 8 \\ 4 & 7 & 2 \end{pmatrix} = 28.$

7. Form matrix: $\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 8 & 11 & 0 & 1 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{pmatrix}.$ Do row operations only, changing

the first three columns into Identity matrix. Then, we get: $\begin{pmatrix} 1 & 0 & 0 & -31 & 9 & -2 \\ 0 & 1 & 0 & 10 & -3 & 1 \\ 0 & 0 & 1 & 4 & -1 & 0 \end{pmatrix}.$

So, the last three columns: $\begin{pmatrix} -31 & 9 & -2 \\ 10 & -3 & 1 \\ 4 & -1 & 0 \end{pmatrix}$ forms the inverse matrix.