hwa0019-0021

1 HW 0019: Rotations, reflections and orthogonal transformations

0019-1: To find a rotation matrix is the same as to find another three vectors so that with $(\sqrt{1/6}, \sqrt{2/6}, 0, \sqrt{3/6})$, they form a orthonormal basis.
First, choose any basis. Here, we choose: $v_1 = (\sqrt{1/6}, \sqrt{2/6}, 0, \sqrt{3/6})$, $v_2 = (0, 1, 0, 0), v_3 = (0, 0, 1, 0), v_4 = (0, 0, 0, 1)$. Then, use Gram-Schmidt Orthonormalization to find orthonormal basis: $x_1 = v_1$;
$w_1 = v_1$; because v_1 is already normal
$x_2 = v_2 - (w_1 \cdot v_2)w_1 = (0, 1, 0, 0) - (\sqrt{1/3})(\sqrt{1/6}, \sqrt{2/6}, 0, \sqrt{3/6}) = (-\sqrt{1/18}, 2/3, 0, -\sqrt{1/6})$;
$w_2 = x_2/|x_2| = (-\sqrt{1/12}, \sqrt{2/3}, 0, -\sqrt{1/4})$; $x_3 = v_3 - (w_1 \cdot v_3)w_1 - (w_2 \cdot v_3)w_2 = (0, 0, 1, 0)$;
$w_3 = x_3/|x_3| = (0, 0, 1, 0)$;
$x_4 = v_4 - (w_1 \cdot v_4)w_1 - (w_2 \cdot v_4)w_2 - (w_3 \cdot v_4)w_3 = (-\sqrt{3}/4, 0, 0, 1/4)$;
$w_4 = x_4/|x_4| = (-\sqrt{3}/2, 0, 0, 1/2)$;
So, $w_1 = v_1, w_2, w_3, w_4$ is the orthonormal basis. they form a matrix:

$$[v_1, w_2, w_3, w_4] = \begin{pmatrix} \sqrt{1/6} & -\sqrt{1/12} & 0 & -\sqrt{3}/2 \\ \sqrt{2/6} & \sqrt{2/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{3/6} & -\sqrt{1/4} & 0 & 1/2 \end{pmatrix}$$

0019-2: $e = (1, 0)$ $v = (\sqrt{3}/2, -1/2)$;
$v^{'} = (-\sqrt{2}/2, \sqrt{2}/2)$;
For v, use method in 0019-1 to find a rotation such that v is the first column.
This matrix is $M = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$.
Similarly, For $v^{'}$, we have a matrix $N = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$.
Observation: $Me = v; Ne = v^{'}$, so $M^{-1}v = e = N^{-1}v^{'}$, so $NM^{-1}v = v^{'}$.
Therefore, matrix $NM^{-1} = \begin{pmatrix} \frac{-\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{-\sqrt{6}-\sqrt{2}}{4} \end{pmatrix}$ is the rotation we want to find.

0019-3: **a**: $\begin{pmatrix} 3/5 & -4/5 & 0 \\ 2\sqrt{2}/5 & 3\sqrt{2}/10 & \sqrt{2}/2 \\ -2\sqrt{2}/5 & -3\sqrt{2}/10 & \sqrt{2}/2 \end{pmatrix}$
b: Check their lengths are 1 and they are orthogonal to each other.
c: For a rotation matrix, $R^{-1} = R^T = \begin{pmatrix} 3/5 & 2\sqrt{2}/5 & -2\sqrt{2}/5 \\ -4/5 & 3\sqrt{2}/10 & -3\sqrt{2}/10 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$.

2 HW0020: Eigenvalues and eigenvectors

0020-1: **a**: characteristic function: $|M - \lambda I| = det \begin{pmatrix} -26-\lambda & 15 \\ -50 & 29-\lambda \end{pmatrix} = 0.$
$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$, so $\lambda_1 = 4, \lambda_2 = -1$.
b: For $\lambda_1 = 4$:
Eigenvector v=(v_1, v_2) satisfies $(M - 4I)v = 0$, that is $\begin{pmatrix} -30 & 15 \\ -50 & 25 \end{pmatrix} v = 0$
Do row operations only to change M-4I to row canonical form:
$N = \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$
So, eigenvector v satisfies Nv=0.
$\Rightarrow v_1 - (1/2)v_2 = 0 \Rightarrow v_1 = (1, 2)$.
Similarly for $\lambda_2 = -1$, $v_2 = (3, 5)$.
c: Matrix with eigenvectors as column is such a matrix C.
So, $C = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$
d: $M = CAC^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$
C is formed by eigenvectors v_1, v_2, A is diagonal matrix with the eigenvaluesλ_1, λ_2 on the diagonal with corresponding order as C is formed by eigenvectors.
d: $e^M = Ce^AC^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} e^4 & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 6e^{-1} - 5e^4 & 3e^4 - 3e^{-1} \\ 10e^{-1} - 10e^4 & 6e^4 - 5e^{-1} \end{pmatrix}$

0020-2: **a**: As in 0020-1, form characteristic functions $M - \lambda I = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 2$.
b: For $\lambda_1 = -1$, $M + I = \begin{pmatrix} -3 & 0 & -3 \\ -3 & 0 & -3 \\ 6 & 0 & 6 \end{pmatrix}$
Do row operations only to change M-4I to row canonical form:
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
So the eigenvectors for -1 are $span\{(1, 0, -1), (0, 1, 0)\}$.
Similarly, for $\lambda_2 = 2$, we have eigenvector (-1,-1,2).
c: $CMC^{-1} = (C^{-1})^{-1}MC^{-1}$

So, By 0020-1, $C^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}; A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

d: $e^{tM} = e^{tC^{-1}AC} = e^{C^{-1}tAC} = C^{-1}e^{tA}C$

3 HW0021: Diagonalization of matrices

NOTE: A matrix is diagonalizable if and only if for all eigenvalues, dimension of the corresponding eigenspace(span of eigenvectors) is equal to the multiplicity of eigenvalues as a root of characteristic function.

0021-1: Step1: find out eigenvalues, $\lambda_1 = 6, \lambda_2 = 3$, both with multiplicity 1.
Step2: find out eigenspaces by methods in 0020-1, $V_1 = span\{v_1 = (3, -5)\}, V_2 = span\{v_2 = (1, -2)\}$. dimension of them are 1.
So, the matrix is diagonalizable.
Step3: invertible matrix A is formed by eigenvectors as columns. $A = \begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix}$

0021-2: Step1: Solve the characteristic functions, there is only one eigenvalue with multiplicity 2.
Step2: Compute dimension of its eigenspaces, which is 1.
Step3: So it is NOT diagonalizable.

0021-3: Comment: One can try to solve the characteristic function, which is a polynomial of degree 3 and find out eigenvalues. And then go through the process in 0021-1 to determine whether it's diagonalizable. But there is no general formula for solutions of polynomial of degree 3. So, sometimes, one may get stuck. That's why there is a hint here.

Step1: write down characteristic function of Z.
Step2: since we know 1 is one eigenvalue, so $\lambda - 1$ is a factor of that degree 3 polynomial. We can divide this polynomial by $\lambda - 1$ to get a polynomial of degree 2. Solve the polynomial of degree 2 to get the other eigenvalues.
Step3: find eigenspace and compare dimension and multiplicity. Then, form invertible matrix if applicable.

0021-4: Nothing different from 0021-3.

Nevertheless, one may try to solve a polynomial of degree 3 directly without the hint.