hwa0022

1 HW 0022: The Spectral Theorem

0022-1: Let $M = \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix}$ be the symmetric matrix corresponding to quadratic form Q.
(a) First find the eigenvalues of M, by solving $\begin{pmatrix} 41-\lambda & 12 \\ 12 & 34-\lambda \end{pmatrix} = (41-\lambda)(34-\lambda) - 144 = 0$.
Eigenvalues $\lambda_1 = 50, \lambda_2 = 25$.
The eigenvector $v_1 = (x_1, y_1)$ of λ_1 is obtained by solving $\begin{pmatrix} 41-50 & 12 \\ 12 & 34-50 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$, which is (4,3). Similarly, another eigenvector $v_2 = (3, -4)$.

Next, DO NOT simply form matrix $R = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$ by eigenvectors as columns as before. The reason is that R satisfies $R^{-1}MR = diagonal$, not $R^tMR = diagonal$, while $Q \circ L_R = Q_{R^tMR}$ in which $R^tMR = diagonal$.

BUT, There is not much left to complete it. Normalize v_1, v_2 to make them unit vectors, that is, of unit length. Then, we have orthonormal vectors $v_1^{'} = (4/5, 3/5), v_2^{'} = (3/5, -4/5)$. Now, it is safe to form $R = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}$ because R is a rotation matrix with $R^{-1} = R^t$.
Remark: For symmetric matrix, eigenvectors of different eigenvalues are automatically orthogonal. This is true only for symmetric matrix.

(b) Both of the eigenvalues are positive, so it should be an ellipse by a rotation of $37°$.