HW 0029: Cholesky Decomposition

0029-1:

a:

Now, we have three ways to address the question.

(1) By spectral theorem, we find eigenvalues and orthonormal eigenvectors, and such eigenvectors will form a matrix A. $D = A^{-1}XA = A^tXA$ is the diagonal matrix. Or $X = ADA^t$. Obviously, every diagonal matrix D with positive diagonal entries can be written as a product $D = BB = BB^t$ of diagonal matrix B. Therefore, $X = ABB^tA^t = (AB)(AB)^t$. So, AB is what we want.

(2) By easier solution in the notes. One do row and column operations simultaneously and change it into Identity matrix.

(3) By Cholesky decompostion, There exists a lower triangular matrix, s.t. $AA^t = X$. We may suppose: $A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$. Then $A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \begin{pmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{pmatrix} = X$. Do matrix multiplication on the left hand side, and compare entries of both sides, then solve unknowns abcdef, one get $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & 2 & 2 \end{pmatrix}$

b:

As in (a.(1)), $X = ABBA^t$. Since $A^tA = I$, $X = ABA^tABA^t$. So we can choose $S = ABA^t$. Therefore, We can conclude that for a symmetric matrix X, the only obstruction for $X = S^2$ is that D has positive diagonal, or all eigenvalues of X are positive.

0029-2:

a:

No. A matrix is positive definite if and only if all eigenvalues are greater than zero.

b:

Yes. A matrix is positive semi-definite if and only if no eigenvalue is negative.

c:

No. CXC^{-1} and X has the same eigenvalues because $det(CXC^{-1} - \lambda I) = det(CXC^{-1} - \lambda CC^{-1}) = det(C[X - \lambda I]C^{-1}) = det(X - \lambda I)$.

d:

Yes. From above.

e:
No. if there is a S, which is symmetric, then S^2 is also symmetric, but the right hand is not symmetric by direct computation.

0029-3:
For **abcd**, methods discussed in 0029-1 apply.
a:
$$A = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ \sqrt{\frac{25}{6}} & \sqrt{\frac{53}{6}} & 0 \\ -\sqrt{\frac{1}{6}} & -\sqrt{\frac{49}{318}} & \sqrt{\frac{36}{53}} \end{pmatrix}$$
b:
$$B = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$
c:
$C = A^t$
d:
$D = B^t$
e:
X is positive definite and symmetric. by 0029-1-b, such S exists.