hwa0032-0033

Section0032: From stirling's Formula to the Central Limit Theorem

0032-1:

a: By definition of f_n, we have $|f_n(x) - x| < (width of bar) = \frac{2}{\sqrt{2n}} \to 0$, as $n \to \infty$.

b: Left hand=$\frac{1}{2n}E[D_{2n}^2]$

What is D_{2n}?

Its distribution models the coin-flip by 2n times.

For k heads and 2n-k tails: Probability=$\frac{1}{2^{2n}}\begin{pmatrix} 2n \\ k \end{pmatrix}$, value of D_{2n} is 2k-2n.

Therefore, $E[D_{2n}^2] = \sum_{k=0}^{k=2n}(2k-2n)^2\frac{1}{2^{2n}}\begin{pmatrix} 2n \\ k \end{pmatrix}$

Right hand=this is an integral of a piecewise constant function, so it can be written as a sum as follows:

At the k-th bar: $f_n(x) = \frac{2k}{\sqrt{2n}}$; $p_n(x) = \frac{\frac{1}{2^{2n}}\begin{pmatrix} 2n \\ n+k \end{pmatrix}}{\frac{2}{\sqrt{2n}}}$

$\int_{-\infty}^{\infty}[f_n(x)]^2[p_n(x)]dx$=$\sum_{k=-n}^{k=n}[\frac{2k}{\sqrt{2n}}]^2[\frac{\frac{1}{2^{2n}}\begin{pmatrix} 2n \\ n+k \end{pmatrix}}{\frac{2}{\sqrt{2n}}}][\frac{2}{\sqrt{2n}}]$=$\sum_{k=0}^{k=2n}[\frac{2k-2n}{\sqrt{2n}}]^2[\frac{\frac{1}{2^{2n}}\begin{pmatrix} 2n \\ k \end{pmatrix}}{\frac{2}{\sqrt{2n}}}][\frac{2}{\sqrt{2n}}]$

Compare both sides: Left hand=Right hand.

Section0033: Piecewise constant processes

0033-1:

Use central limit theorem and the fact that $\frac{X_t^{(N)}}{\sqrt{t}}$ is approximately normally distributed.

a:

$(\sqrt{t}\frac{X_t^{(N)}}{\sqrt{t}})^4$, so the function $f(x) = t^2x^4$

$\lim_{N\to\infty} E[(X_t^{(N)})^4] = \int_{-\infty}^{\infty} f(x)e^{-\frac{x^2}{2}}dx = 3t^2$

b:

The function is $f(x) = e^{8\sqrt{t}x}$

$\lim = \int_{-\infty}^{\infty} e^{8\sqrt{t}x}e^{-\frac{x^2}{2}}dx = e^{32t}$

c:
The function is $f(x) = (e^{8\sqrt{t}x} - e^{16})_+$
$\int_{-\infty}^{\infty} (e^{8\sqrt{t}x} - e^{16})_+ e^{-\frac{x^2}{2}} dx = e^{32t}\phi(8\sqrt{t} - \frac{2}{\sqrt{t}}) - e^{16}\phi(-\frac{2}{\sqrt{t}})$