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Section0034 Vector Fields and ODEs

0034-1:
Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $A = \begin{pmatrix} 3 & 0 & 9 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}$. Then the set of linear differential equations is replaced by a matrix differential equation of $v' = \frac{dv}{dt} = Av$. As we know, $(ce^{kt})' = kce^{kt}$, where k,c are constant. So, we may conclude that $v(t) = e^{At}w$, where A is the matrix, w is a constant vector, is a solution we want. Let $t = 0$, $w = v(0) = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$
Next, we compute e^{At}. First, we write $A = BDB^{-1}$, in which B is the matrix of eigenvectors of A, D is diagonal matrix of eigenvalues of A. Then, $e^{At} = Be^{Dt}B^{-1}$ by results in previous homework. Last, it is easy to compute e^{Dt} since Dt is a diagonal matrix, for which exponential of Dt is equal to the diagonal matrix with diagonal entries the exponential of Dt's diagonal entries.
The answer is $v = \begin{pmatrix} 11e^{3t} - 3 \\ 3e^{2t} + 2 \\ 1 \end{pmatrix}$

0034-2:
from 0034-1, we know that z=1 forever. So, we may view $9 = 9z, -4 = -4z$. Therefore, actually, conditions for x, y in 0034-2 is the same as conditions in 0034-1. Thus, the answer should be the same.
$\begin{pmatrix} x = 11e^{3t} - 3 \\ y = 3e^{2t} + 2 \end{pmatrix}$

0034-3:
Transforming this geometric question into mathematical terms as follows:
Let $c(t) = (x(t), y(t))$, then $(x', y') = (3x + 9, 2y - 4)$ and $x(0) = 8, y(0) = 5$
We have already had the answer for the differential equations.

0034-4:
$-\nabla f(x, y) = -(6x + 4y + 22, 6y + 4x + 18)$
Let $c(t) = (x(t), y(t))$,
So, you want to solve the equations:

$x^{'} = -6x - 4y - 22$
$y^{'} = -6y - 4x - 18$
$x(0) = 1, y(0) = 1$
As above, insert a new variable, the corresponding equations are:
$x^{'} = -6x - 4y - 22z$
$y^{'} = -6y - 4x - 18z$
$z^{'} = 0$
$x(0) = 1, y(0) = 1, z(0) = 1$

The answer is $c(t) = (3e^{-10t} + e^{-2t} - 3, 3e^{-10t} - e^{-2t} - 1)$

Section0036 The multi-variable chain rule

0036-1:

a: $f_s^{'}(s,t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $f_t^{'}(s,t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

So, $f^{'}(1,1) = (f_s^{'}(1,1), f_t^{'}(1,1)) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$

b: $f(1,1) = \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix}$

$g^{'}(x,y,z) = (g_x^{'}, g_y^{'}, g_z^{'})$

and $g^{'}(f(1,1)) = g^{'}(2,0,7) = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 0 \\ 0 & 14 & 0 \end{pmatrix}$

c: Multiplying results of **a** and **b**. The answer is $\begin{pmatrix} 2 & -2 \\ 2 & -1 \\ 2 & 2 \\ 14 & -14 \end{pmatrix}$

d: $g \circ f(s,t) = (\sin(s^2 - t^2), 2s - t - 6, 2s + 2t + s^2 - 2st + t^2, s^3 + 6s^2 - st^2 - 6t^2)$.

e: $(g \circ f)^{'}(s,t) = \begin{pmatrix} 2 & -2 \\ 2 & -1 \\ 2 & 2 \\ 14 & -14 \end{pmatrix}$

Section0037 Constrained optimization

Theorem: The condition for f(x,y) taking extreme value at (x_0, y_0) under constraint $g(x,y) = 0$ is $\bigtriangledown f(x_0, y_0) = \lambda \bigtriangledown g(x_0, y_0)$ and $g(x_0, y_0) = 0$.
remark1: the result may extend to n variable function without change.
remark2: nothing can say if it happens $\bigtriangledown f = \bigtriangledown g = 0$. for those points, one

need to check directly by computing their values.

0037-1:
In this question, $f(x, y) = 2x - 7y$, $g(x, y) = x^4 + y^4 - 1$
$\bigtriangledown f = (2, -7)$, $\bigtriangledown g = (4x^3, 4y^3)$
So, we have to solve the following equations:
$$x^4 + y^4 - 1 = 0$$
$$2 = 4\lambda x^3$$
$$-7 = 4\lambda y^3$$
Kill λ in 2nd and 3rd equations, and we have:
$$x = \pm(-2/7)^{1/3}[1 + (2/7)^{4/3}]^{-1/4}$$
$$y = \pm[1 + (2/7)^{4/3}]^{-1/4}$$
By direct compute values of f at the above points $\Longrightarrow (x, y, \lambda) = ((2/7)^{1/3}[1 + (2/7)^{4/3}]^{-1/4}, -[1 + (2/7)^{4/3}]^{-1/4}, 7/4[1 + (2/7)^{4/3}]^{1/4})$
and maximal of f is $[2(2/7)^{1/3} + 7][1 + (2/7)^{4/3}]^{-1/4}$

0037-2:
The same method as above. It is almost always the case that there may more than one solutions, one need compute and compare the corresponding values to determine the answer. The answer is $[2(-2/9)^{1/5} - 9][1 + (2/9)^{6/5}]^{1/6}$

0037-3:
a: $(a_n, b_n) = ((1/2)^{\frac{1}{2n}}, (1/2)^{\frac{1}{2n}})$
b: (1,1)

0037-4:
Condition(1): invest 10.
$g_1 = b + p + q - 10 = 0$
Condition(2): 7.7 percent return.
$g_2 = 1.04b + 1.07p + 1.12q - (1 + 0.077)(b + p + q) = 0$
Target: minimize variance $V = Var[bB + pP + qQ] = Var[bB] + Var[pP] + Var[qQ] + 2Cov[pP, qQ] = 0.2p^2 + 0.3q^2 + 0.3pq$

Then, we solve set of equations:
$$g_1 = 0$$
$$g_2 = 0$$
$$\bigtriangledown V = \lambda_1 \bigtriangledown g_1 + \lambda_2 g_2$$
we have $(b, p, q) \approx (6.21, -1.34, 5.13)$
This is the answer if we assume share taken values in continuous numbers.

A long remark:
(1)For mathematical rigorous, the above work is not at all complete. When one consider extreme value problems, what is the behavior of function f on the boundary is very important. The above method only guarantee to find "interior" points where f takes extreme values. An example: f(x)=x at the interval

[0,1]. By the above method, I know that there is no extreme values, but if we also check the boundary, we find that there are maximal and minimal values.
Ok, by awareness of checking boundary, what if the domain of f is infinite? for in 0037-4, (b,p,q)$\in R^3$ or say $\in Z^3$. The answer is we also need to check the limit behavior of f as its variable such as b,p,q tends to infinite. In the above case, it is obvious that when b,p,q tends to infinite, f, or V precisely, tends to infinite. Since we want to find minimal, this causes no problem.
If you encounter a more complicated function V, do not forget to discuss its finite boundary or infinite limit situation.
(2)Some one may wish to find integer solution, not continuous one. This is no easier problem as above.
A natural approach is to take the closest integer for every variable once we have a decimal number by the above method. Unfortunately, this doesn't work in general case, and is dangerous! You make an assumption that the integer closest to point that takes the extreme value will itself take a extreme value in the set of all integer numbers. This is an unjustified assumption. It is not true in general case.
(3) In Q0037-4, you may compute all nearby integer points to determine an integer solution, which of course will not satisfy 7.7 return condition, but which minimize variance. For some bad function, not as good as quadratic function, the minimal integer point may not even be a nearby point.