# Answers for first midterm

**I: Definitions**

**a**: A set in $R^n$ is compact if it is both closed and bounded.

**b**: A function is injective if $\forall a, a^{'} \in X$, and $a \neq a^{'}$ implies $f(a) \neq f(a^{'})$

**c**: b is the infimum of S if b is a lower bound of S, and $\forall$ lower bound a for S, we have $a \leq b$

**d**: A map L is linear if both of the following hold:

- for all $v, v^{'} \in V$, $F(v + v^{'}) = F(v) + F(v^{'})$
- for all scalar c and $v \in V$, F(cv)=cF(v)

**e**: $v \cdot w = a_1 b_1 + a_2 b_2 + ... + a_n b_n$

**II: True or False**

**a**: T
**b**: F
**c**: T
**d**: T
**e**: F

**III: Computations**

**1**

Let $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$. we have $\frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt = \Phi(-x)$. Let x=0, we get $\int_{0}^{\infty} e^{-\frac{-x^2}{2}} dx = \sqrt{2\pi}\Phi(0)$
$\int_{0}^{\infty} (2x - 5) e^{-\frac{x^2}{2}} dx$
$= \int_{0}^{\infty} 2x e^{-\frac{x^2}{2}} dx - 5 \int_{0}^{\infty} e^{-\frac{x^2}{2}} dx$
$= \int_{0}^{\infty} 2x e^{-\frac{x^2}{2}} dx - 5\sqrt{2\pi}\Phi(0)$
$= \int_{0}^{\infty} e^{-\frac{x^2}{2}} d(x^2) - 5\sqrt{2\pi}\Phi(0)$

$= -2\int_0^\infty e^{-\frac{x^2}{2}} d(-\frac{x^2}{2}) - 5\sqrt{2\pi}\Phi(0)$
$= -2e^{-\frac{x^2}{2}}|_0^\infty - 5\sqrt{2\pi}\Phi(0)$
$= 2 - 5\sqrt{2\pi}\Phi(0)$

**2**

$\binom{10+7}{7} - \binom{9+7}{7} = \binom{17}{7} - \binom{17}{7} = \frac{7\times16!}{7!10!}$

**3**

$$L_M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \\ 4 & 0 & -5 \end{pmatrix} \begin{pmatrix} v \\ w \\ x \end{pmatrix}$$

**4**

**a,b**

By fact on page 40 of section0004, $f(x) - 0 = f^{(3)}(t)\frac{x^3}{3!}$ for some t. Here p(x)=0 by condition in this question. Thus, $f(-1) = f^{(3)}(t)\frac{(-1)^3}{3!} = -\frac{1}{6}f^{(3)}(t)$. Because $6 \le f^{(3)}(t) \le 12$, we get $-2 \le f(-1) \le -1$

**5**

Let $y = (x^2+1)^x$, then $\ln y = x\ln(x^2+1)$
$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln(x^2+1)$
$\frac{y'}{y} = \ln(x^2+1) + x\frac{2x}{(x^2+1)}$
$\frac{y'}{y} = \ln(x^2+1) + \frac{2x^2}{(x^2+1)}$
$y' = y[\ln(x^2+1) + \frac{2x^2}{(x^2+1)}]$
$y' = (x^2+1)^x[\ln(x^2+1) + \frac{2x^2}{(x^2+1)}]$

**6**

35

**7**

$e^{\frac{\ln 2}{2} + i\frac{\pi}{4}}$
$= e^{\frac{\ln 2}{2}} e^{i\frac{\pi}{4}}$

$= \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
$= 1 + i$

**8**

Let $M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Then $L_M(x,y) = \begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ bx+dy \end{pmatrix}$
So, $L_M(x,y)\cdot(x,y) = (ax+by, bx+dy)\cdot(x,y) = (ax+by)x + (bx+dy)y = ax^2 + 2bxy + dy^2$
Comparing coefficients, we have (a,b,d)=(1,2,1)
$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$