I. Definitions: Complete the following sentences.

1. (3 pts.) Let a_n and b_n be two sequences. We say that a_n and b_n are **asymptotic** if...

$$\frac{a_n}{b_n} \to 1$$

3

2. (3 pts.) Let $X : \Omega \times [0, \infty) \to \mathbb{R}$ be a function, and assume, for all $t \in [0, \infty)$, that $X_t := X(\bullet, t)$ is a PCRV. Then X is a **Δt-piecewise constant process** if...

for all $n \in \{0, 1, 2, \ldots, \}$

3

for all $u, t \in [n(\Delta t), (n+1)(\Delta t))$

Such that $X(\cdot, u) = X(\cdot, t)$

3. (3 pts.) Let X be a PCRV, and let $A \subseteq [0, 1]$ be finite union of intervals. Assume that A has positive size. Then $\mathrm{E}[X|A] = \cdots$.

$$\frac{1}{|A|} \int_A X(w)\, dw$$

3

4. (3 pts.) Let X be a PCRV, and let $\mathcal{P}$ be a partition of Ω by finite unions of intervals. Then $\mathrm{E}[X|\mathcal{P}]$ is the PCRV defined by the rule: For all $\omega \in \Omega$, if $\omega \in B \in \mathcal{P}$ (and if B is not of zero size), then $(\mathrm{E}[X|\mathcal{P}])(\omega) = \cdots$

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$$E[X|B]$$

5. (3 pts.) A PCRV X is said to be **standard** if...

$$E[X] = 0$$

$$Var[X] = 1$$

3

II. True or False. (No partial credit.)

T ~~a~~. (2 pts.) If two PCRVs have the same distribution, then they have the same variance.

F b. (2 pts.) If two PCRVs have the same distribution, then they are independent.

F ~~c~~. (2 pts.) Let $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$ be the standard Brownian motion approximation. Then $\mathrm{Var}[X_3^{(n)}] \to 9$, as $n \to \infty$.

$\frac{X_3^{(n)}}{\sqrt{3}}$

T ~~d~~. (2 pts.) Let $f(t)$ be the Fourier transform of the distribution of a PCRV. Then $f(0) = 1$.

T ~~e~~. (2 pts.) For any symmetric, positive definite matrix $A \in \mathbb{R}^{2\times 2}$, there exists a matrix $B \in \mathbb{R}^{2\times 2}$ such that $BB^t = A$.

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I.

II.

III.1.

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3. Computations. Some of your answers may involve Φ, the cumulative distribution function of the standard normal distribution. (Answers typically must be exactly correct. No partial credit, except in unusual situations. No need to simplify arithmetic.)

1. Let X be a binary PCRV such that $\Pr[X = 6] = 0.7$ and $\Pr[X = -4] = 0.3$. Let $f(t)$ be the Fourier transform of the distribution of X.

a. (5 pts.) Compute $f''(0)$.

$$f(t) = 0.7\, e^{-it6} + 0.3\, e^{it4}$$

$$f'(t) = (-i6)\, 0.7\, e^{-it6} + (i4)\, 0.3\, e^{it4}$$

$$f''(t) = (-i6)^2\, 0.7\, e^{-it6} + (i4)^2\, 0.3\, e^{it4} = -25.2\, e^{-it6} - 4.8\, e^{it4}$$

$$f''(0) = -25.2 + 4.8 = -30$$

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b. (5 pts.) Compute $E[X^2]$.

$$E[X^2] = 36 \times 0.7 + 16 \times 0.3$$

$$= 25.2 + 4.8$$

$$= 30$$

5

~~2.~~ (20 pts.) Let $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$ be a the standard Brownian motion approximation. For all integers $n \geq 1$, let $Y_n := X_4^{(n)}$. Compute $\lim_{n\to\infty} \mathrm{E}[(e^{Y_n} - e^8)_+]$.

$$\lim_{n\to\infty} E[(e^{X_4^{(n)}} - e^8)_+] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (e^{\sqrt{4}x} - e^8)_+ e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_4^{\infty} (e^{2x} - e^8) e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_2^{\infty} e^2 e^{-\frac{x^2}{2}}\,dx - \frac{1}{\sqrt{2\pi}}\int_4^{\infty} e^8 e^{-\frac{x^2}{2}}\,dx$$

$$= e^2\,\Phi(-2) - e^8\,\Phi(-4)$$

$$e^{\sqrt{4}x} \geq e^8 \Rightarrow \sqrt{4}x \geq 8 \Rightarrow x \geq \frac{8}{2} = 4$$

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3. (10 pts.) Using Stirling's formula ($n! \sim \sqrt{2\pi n}(n/e)^n$), find constants C, k and b such that

$$\binom{2n}{n} \sim C(n^k)(b^n)$$

(For full credit, be sure to state the values of all three: C, k *and* b.)

$$\frac{(2n)!}{n!\,n!} \sim \frac{\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}} = \sqrt{\frac{1}{\pi}}\, n^{-\frac{1}{2}}\, 4^n$$

$$\therefore C = \sqrt{\frac{1}{\pi}},\quad k = -\frac{1}{2},\quad b = 4$$

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4. (10 pts.) Suppose $X_1, \ldots, X_{100}$ are iid PCRVs, and let $A = X_1 + \cdots + X_{100}$. If $\mathrm{SD}[A] = 0.2$, compute $\mathrm{SD}[X_1]$.

$$\mathrm{Var}[A] = 0.04$$

$$\mathrm{Var}[X_1] = \frac{0.04}{160} = 0.0004$$

$$\mathrm{SD}[X_1] = \sqrt{0.0004} = 0.02$$

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5. (10 pts.) Let $C_1, C_2, \ldots$ be standard PCRVs, and assume that $C_1, C_2, \ldots$ are iid. For all integers $n \geq 1$, let $R_n := 2 + (0.2)(C_1 + \cdots + C_n)/\sqrt{n}$. Compute $\lim_{n\to\infty} \mathrm{E}[e^{R_n}]$.

$$\lim_{n\to\infty} E[e^{R_n}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{0.2x+2}\, e^{-\frac{x^2}{2}}\, dx$$

$$= e^{\frac{(0.2)^2}{2}+2}$$

$$= e^{2.02}$$

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6. (10 pts.) Let $X^{(1)}, X^{(2)}, X^{(3)}, \ldots$ be the standard Brownian motion approximation. Compute $\lim_{n\to\infty} \mathrm{E}[(X_9^{(n)})^8]$.

$$\lim_{n\to\infty} E\left[\left(\frac{X_9^{(n)}}{\sqrt{9}}\right)^8 (\sqrt{9})^8\right] = \lim_{n\to\infty} E\left[\left(\frac{X_9^{(n)}}{3}\right)^8 3^8\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 3^8 \cdot x^8 \cdot e^{-\frac{x^2}{2}}\, dx$$

$$= 3^8 \cdot 7 \cdot 5 \cdot 3 \cdot 1$$

$$= 3^9 \cdot 7 \cdot 5$$

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7. (5 pts.) Let $X := \begin{cases} 1, & \text{if } 0 \le \omega \le 1/2; \\ 3, & \text{if } 1/2 < \omega \le 1, \end{cases}$ and let $Y := \begin{cases} 100, & \text{if } 0 \le \omega < 1/2; \\ -100, & \text{if } 1/2 \le \omega \le 1. \end{cases}$

Calculate $\mathrm{Corr}[X, Y]$.

$E[XY] = 100 \times \frac{1}{2} - 300 \times \frac{1}{2} = 50 - 150 = -100$

$E[X] = 1 \times \frac{1}{2} + 3 \times \frac{1}{2} = \frac{1+3}{2} = 2$

$E[Y] = 100 \times \frac{1}{2} - 100 \times \frac{1}{2} = 50 - 50 = 0$

$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y] = -100$

$SD[X] = 0.5 \times 2 = 1$

$SD[Y] = 0.5 \times 200 = 100$

$\mathrm{Corr}[X, Y] = \frac{\mathrm{Cov}[X, Y]}{SD[X]\,SD[Y]} = \frac{-100}{100} = -1$

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