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Graded G-sets, Symmetric Powers of Permutation Modules, and the Cohomology of Wreath Products

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With best wishes to Mark Mahowald for his 60th birthday

Abstract. The Poincaré series of the cohomology ring of a wreath product of groups may be rather easily computed by a combinatorial procedure in terms of the Poincaré series of the groups in the wreath product. The method is applied to the cohomology of the Sylow 2-subgroup of $GL(4,2)$, and also gives the decomposition of symmetric powers of permutation modules.

Our main result at the end of this paper is a formula for the Poincaré series of the cohomology of a wreath product of groups $F \wr G$ with $\mathbb{F}_p$ coefficients, expressed in terms of the Poincaré series of F and the subgroups of G, and the Möbius function for the poset of subgroups of G. This formula makes it comparatively easy to compute the Poincaré series for some quite complicated groups, and as an example we use it to derive the Poincaré series for the cohomology of the Sylow 2-subgroup of $GL(4,2) \cong A_8$, which may be expressed as a wreath product.

Our main input for the cohomology is the result proved by Nakaoka [**8**] which gives the algebra structure of $H^*(F \wr G, \mathbb{F}_p)$. Nakaoka's result could hardly be more transparent as a description of this structure. On the other hand, to extract the Poincaré series from his description requires a little bit of combinatorial work, and it is this work which we perform here. The same problem has previously been treated by Bogačenko [**3**] who obtained our formula for the Poincaré series in the case where G is cyclic of order p.

To obtain the cohomology formula one needs to study certain constructions to do with permutation modules, and these in turn depend on the corresponding constructions with G-sets. Accordingly, we start with the G-sets and as a special case of what we do, we define the symmetric powers of a G-set. These are exactly what one would imagine them to be, and it may help in understanding the relevance

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of our general formulation to guess how the symmetric powers of a set might be defined. We then turn G-sets into permutation modules and as a corollary of the general theory obtain a description of the symmetric powers of permutation modules. The significance of this is that we obtain the full module structure of the symmetric powers, but in particular we can deduce the (known) formula for the Poincaré series for the ring of invariants in the symmetric algebra on a permutation module.

Since this paper was first written it has come to my attention that there is related material in the work of Liulevicius and Özaydin [**6,7**]. They consider the symmetric powers of G-sets and obtain a formula for their structure in the special case that G is cyclic. Their formula involves the Möbius function of number theory, which in this case coincides with the Möbius function for the poset of subgroups of the cyclic group. Several ideas needed for this work are present in their papers, and I wish to acknowledge their priority at this point without making special reference later on. For the application to the cohomology of wreath products made here the generality of their formula is insufficient, both because they restrict the structure of G and because one needs to consider more general G-sets than symmetric powers.

1. Graded G-sets

By a *graded set* we mean a set Ψ partitioned as

$$\Psi = \Psi(0) \cup \Psi(1) \cup \Psi(2) \cup \cdots$$

We will refer to $\Psi(i)$ as the set of *elements in degree* i. A *graded* G*-set* is a graded set in which each of the components $\Psi(i)$ is a G-set. We denote by $B(G)$ the Burnside ring of finite G-sets with rational coefficients (c.f. [**2**]). This is a $\mathbb{Q}$-vector space with the inequivalent transitive G-sets G/H as a basis. Assuming that each of the sets $\Psi(i)$ is finite, we form what (by an extension of usual terminology) we may call the *Poincaré series* of a graded G-set Ψ:

$$P_\Psi(t) = \sum_{i=0}^{\infty} \Psi(i) t^i.$$

This is a formal power series in the variable t with coefficients in $B(G)$. Our first aim is to show how this series may be computed.

Burnside [**4**, **p.238**] introduced a homomorphism $m : B(G) \to \mathrm{Map}(\mathcal{C}, \mathbb{Q})$ where $\mathcal{C}$ is the set of conjugacy classes of subgroups of G and $\mathrm{Map}(\mathcal{C}, \mathbb{Q})$ is the algebra of $\mathbb{Q}$-valued functions on $\mathcal{C}$. In fact we will usually work with a set of representatives for the conjugacy classes of subgroups and use $\mathcal{C}$ to denote also this set of representatives. We write an element of $\mathrm{Map}(\mathcal{C}, \mathbb{Q})$ as a list of its values on the elements of $\mathcal{C}$: $(a_J)_{J\in\mathcal{C}}$ where $a_J \in \mathbb{Q}$ and J ranges over a set of representatives of the conjugacy classes of subgroups of G. The homomorphism which Burnside introduced is defined by $m(\Omega) = (|\Omega^J|)_{J\in\mathcal{C}}$ where the superscript J denotes fixed points, and he showed that this map is injective. Evidently m extends to a map of power series $m : B(G)[[t]] \to \mathrm{Map}(\mathcal{C}, \mathbb{Q})[[t]]$. Given a graded G-set Ψ, we define for each subgroup J of G a power series $f_J(t)$ with integer coefficients by

$$f_J(t) = \sum_{i=0}^{\infty} |\Psi(i)^J| t^i.$$

Then clearly

$$m(P_\Psi) = (f_J)_{J\in\mathcal{C}}.$$

This allows us to derive a formula for P_Ψ in the following way. We let $e_J \in B(G)$ be the idempotent for which $m(e_J) = 1$ at J and 0 elsewhere (see [**5**] and [**11**]). We have immediately:

(1.1) PROPOSITION.

$$P_\Psi = \sum_{\substack{J\le G \\ \text{up to conjugacy}}} f_J e_J.$$

PROOF. Since m is injective it suffices to check that both sides are sent to the same element of $B(G)[[t]]$ by m. That this is so is immediate from the definitions.□

There is an explicit formula for these idempotents which is due to Gluck [**5**] and Yoshida [**11**]:

$$e_J = \frac{1}{|N_G(J)|}\sum_{K\le J} |K|\cdot G/K\cdot \mu(K,J).$$

Here μ is the Möbius function of the partially ordered set of subgroups of G. We may combine this formula with the statement of 1.1 to give

(1.2) PROPOSITION.

$$P_\Psi = \sum_{K\le J} \frac{G/K\,\mu(K,J)\,f_J}{|G:K|}.$$

PROOF. Here the sum is over all pairs of subgroups $K \le J$. The formula is immediate from what we had previously, since on summing over all subgroups instead of over conjugacy classes we have

$$P_\Psi = \sum_J \frac{f_J\,e_J}{|G:N_G(J)|} = \sum_J \frac{f_J}{|G:N_G(J)|}\cdot\frac{1}{|N_G(J)|}\sum_{K\le J}|K|\cdot G/K\cdot\mu(K,J).$$

□

As an example of this we take $G = C_2\times C_2$, in which case on writing out explicitly the values of the Möbius function in 1.2 we obtain

$$\begin{aligned} P_\Psi =& f_G\left(G/G - \frac{(G/A+G/B+G/C)}{2} + \frac{2\cdot G/1}{4}\right)\\ &+ f_A\left(\frac{G/A}{2}-\frac{G/1}{4}\right) + f_B\left(\frac{G/B}{2}-\frac{G/1}{4}\right) + f_C\left(\frac{G/C}{2}-\frac{G/1}{4}\right) \qquad (1.3)\\ &+ f_1\left(\frac{G/1}{4}\right)\end{aligned}$$

Quite how we compute the power series f_J depends on Ψ, but for a certain type of naturally occuring graded G-set we show how this may be done.

Suppose we are given a graded set Ξ and a finite G-set Ω. We may form the G-set $\mathrm{Map}(\Omega, \Xi)$ of functions from Ω to Ξ. This becomes a G set via the action of G on Ω. If $\phi \in \mathrm{Map}(\Omega, \Xi)$ we define its degree to be

$$\deg \phi = \sum_{\omega \in \Omega} \deg \phi(\omega).$$

Then $\mathrm{Map}(\Omega, \Xi)$ becomes a graded G-set.

For example, suppose that Ξ has precisely one element in each degree, so we may take $\Xi = \mathbb{N}$ to be the natural numbers $\{0, 1, 2, 3, \ldots\}$, and suppose $\Omega = \{a, b, c\}$. Then a function $\phi : \Omega \to \mathbb{N}$ may be written as $a^r b^s c^t$ where $\phi(a) = r$, $\phi(b) = s$ and $\phi(c) = t$, so that the elements of $\mathrm{Map}(\Omega, \mathbb{N})$ of degree n may be identified as the monomials of degree n in the elements of Ω. Thus we produce for each finite G-set Ω and natural number n the G-set $\mathrm{Map}(\Omega, \mathbb{N})(n)$, which we term the *nth symmetric power of* Ω. It is convenient to denote this G-set by $S^n(\Omega)$.

In general the power series f_J for $\mathrm{Map}(\Omega, \Xi)$ are computed as follows.

(1.4) Proposition. *Let Ξ be a graded set with Poincaré series f, and let Ω be a finite G-set.*

1. *For each subgroup $J \le G$ let f_J be the power series*

$$f_J(t) = \sum_{i=0}^{\infty} |(Map(\Omega, \Xi)(i)^J| t^i.$$

Then

$$f_J(t) = f(t^{|\Omega_1|}) \cdots f(t^{|\Omega_n|})$$

where $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$ is the decomposition of Ω into J-orbits.

2. *With the f_J given as in (1), the Poincaré series of the graded G-set $Map(\Omega, \Xi)$ is*

$$P_{\mathrm{Map}(\Omega,\Xi)} = \sum_{K \le J} \frac{G/K \, \mu(K, J) \, f_J}{|G : K|}$$

in $B(G)[[t]]$.

Proof. (1) One sees immediately that $(\mathrm{Map}(\Omega, \Xi))^J$ consists of those functions ϕ which are constant on the orbits of J on Ω. Such functions biject with functions on the orbit space $\overline{\phi} : J \backslash \Omega \to \Xi$, where for each $\phi : \Omega \to \Xi$ constant on J-orbits, the associated map $\overline{\phi}$ is defined by $\overline{\phi}(\Omega_i) = \phi(\omega_i)$, where $\omega_i \in \Omega_i$. The degree of such a map ϕ is

$$\deg \phi = \sum_{\omega \in \Omega} \deg \phi(\omega) = \sum_{i=1}^{n} |\Omega_i| \deg \phi(\omega_i).$$

We should therefore define

$$\deg \overline{\phi} = \sum_{i=1}^{n} |\Omega_i| \deg \overline{\phi}(\Omega_i)$$

and now with this definition of degree $\mathrm{Map}(J \backslash \Omega, \Xi)$ has the same Poincaré series as $(\mathrm{Map}(\Omega, \Xi))^J$.

For each orbit Ω_i the functions $\{\Omega_i\} \to \Xi$ with domain the one point set $\{\Omega_i\}$ and degrees multiplied by $|\Omega_i|$ as above, have Poincaré series $f(t^{|\Omega_i|})$. Since an arbitrary function in $\mathrm{Map}(J\backslash\Omega, \Xi)$ may be regarded as an n-tuple $(\overline{\phi}(\Omega_1), \ldots, \overline{\phi}(\Omega_n))$ the Poincaré series we require is the product of the $f(t^{|\Omega_i|})$.

(2) This is a restatement of formula 1.2. $\square$

We point out also a generalization of this result in which Ω is itself taken to be a graded G-set, and if $\phi \in \mathrm{Map}(\Omega, \Xi)$ we define

$$\deg \phi = \sum_{\omega \in \Omega} \deg(\omega) \cdot \deg \phi(\omega).$$

Then a similar result to 1.4 holds, the difference being that we should take

$$f_J = f(t^{d_1|\Omega_1|}) \cdots f(t^{d_n|\Omega_n|})$$

where d_i is the degree of the elements in Ω_i. We leave the proof of this modification to the reader.

As an example, let $G = C_2 \times C_2$ act regularly on a set Ω of size 4. We will compute the decomposition into orbits of the symmetric powers $S^n(\Omega)$. We take $\Xi = \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ so that $f(t) = \frac{1}{1-t}$. As well as the identity subgroup and the whole group, there are three further subgroups of G, which have size 2 and which we denote A, B and C. They each have two orbits on Ω of size 2. Now by Proposition 1.4,

$$f_1(t) = \frac{1}{(1-t)^4}, \qquad f_A(t) = f_B(t) = f_C(t) = \frac{1}{(1-t^2)^2}, \qquad f_G(t) = \frac{1}{1-t^4}.$$

Using the previous formula 1.3 for P_Ψ with $\Psi = \mathrm{Map}(\Omega, \mathbb{N})$ we obtain

$$\begin{aligned}
P_{\mathrm{Map}(\Omega,\mathbb{N})} =& f_G\, G/G \\
&+ \left(\frac{-f_G}{2} + \frac{f_A}{2}\right) G/A + \left(\frac{-f_G}{2} + \frac{f_B}{2}\right) G/B + \left(\frac{-f_G}{2} + \frac{f_C}{2}\right) G/C \\
&+ \left(\frac{f_G}{2} - \frac{f_A + f_B + f_C}{4} + \frac{f_1}{4}\right) G/1 \\
=& \frac{G/G}{1-t^4} + \left(\frac{1}{(1-t^2)^2} - \frac{1}{1-t^4}\right) \frac{G/A + G/B + G/C}{2} \\
&+ \left(\frac{1}{1-t^4} - \frac{3}{2(1-t^2)^2} + \frac{1}{2(1-t)^4}\right) \frac{G/1}{2} \\
=& \frac{1}{1-t^4} G/G \\
&+ \frac{t^2}{(1-t^2)(1-t^4)} (G/A + G/B + G/C) \\
&+ \frac{t - t^2 + 3t^3 - t^4}{(1-t)^2 (1-t^2)(1-t^4)} G/1.
\end{aligned}$$

Using the power series expansions

$$\frac{1}{1-t^4} = 1 + t^4 + t^8 + t^{12} + t^{16} + \cdots$$
$$\frac{t^2}{(1-t^2)(1-t^4)} = t^2 + t^4 + 2t^6 + 2t^8 + 3t^{10} + 3t^{12} + 4t^{14} + 4t^{16} + \cdots$$
$$\frac{t - t^2 + 3t^3 - t^4}{(1-t)^2(1-t^2)(1-t^4)} = t + t^2 + 5t^3 + 7t^4 + 14t^5 + 18t^6 + 30t^7 + 38t^8 + \cdots$$

we obtain, for example, that the 8th symmetric power $S^8(\Omega)$ is the union of 38 regular orbits, 2 orbits with each of the subgroups A, B and C as stabilizer, and 1 fixed point.

2. Symmetric powers of permutation modules

The type of calculation performed in the last section allows us to find very easily the complete module structure of the symmetric algebra on a permutation module. The answers are independent of the ground ring we work over, which may be a field of characteristic p, or characteristic 0, or some quite different ring. This feature is not really surprising since our answer for the module structure is given as a decomposition into a direct sum of permutation modules, and this type of decomposition is a property of G-sets rather than of modules.

We fix a commutative ground ring R and consider RG-modules. For any RG-module V we let $S^\bullet(V) = \bigoplus_{i=0}^{\infty} S^i(V)$ be the symmetric algebra on V. In particular we will be interested in the case when $V = R\Omega$ is the permutation module whose basis is a G-set Ω. The symmetric power $S^i(R\Omega)$ is now the module whose basis consists of the monomials in Ω of degree i. Since these monomials may be identified with $\mathrm{Map}(\Omega, \mathbb{N})(i) = S^i(\Omega)$ the decomposition of $\mathrm{Map}(\Omega, \mathbb{N})$ into orbits immediately gives a decomposition of $S^\bullet(R\Omega)$ as a direct sum of transitive permutation modules.

For example, suppose that $G = C_2 \times C_2$ acts regularly on a set Ω of size 4, as in Section 1. Then by our previously worked example we have

$$S^8(R\Omega) = R \oplus (R\uparrow_A^G \oplus R\uparrow_B^G \oplus R\uparrow_C^G)^2 \oplus (R\uparrow_1^G)^{38}$$

using as before the notation A, B, C for the three subgroups of size 2.

It is convenient to make a Poincaré series out of this, but we have to replace the Burnside ring we used in Section 1 by some different ring of coefficients. We will use the *Green ring* $A(G)$ (c.f. [**2**]) which we will take to be the $\mathbb{Q}$-vector space with the set of isomorphism classes of indecomposable RG-modules as a basis. Given an RG-module M which is a finite direct sum of indecomposables $M \cong M_1 \oplus \cdots \oplus M_n$ we denote by M also the linear combination of basis elements $M_1 + \cdots + M_n \in A(G)$. In order that the symbol M should represent a well-defined element of $A(G)$ we should assume that the Krull-Schmidt theorem holds for RG-modules (as happens if R is a field or a complete discrete valuation ring), and when mentioning $A(G)$ we will always implicitly make this assumption.

If M happens to be a graded RG-module $M = M(0) \oplus M(1) \oplus M(2) \oplus \cdots$ such that each $M(i)$ is a finite direct sum of indecomposables we may associate a power series

$$P_M(t) = \sum_{i=0}^{\infty} M(i)t^i \in A(G)[[t]]$$

and by analogy with other related series we will call this the *Poincaré series* of M. We will perform this construction in the particular case that we have a graded G-set Ψ, when we may form the graded module

$$R\Psi = R[\Psi(0)] \oplus R[\Psi(1)] \oplus R[\Psi(2)] \oplus \cdots .$$

(2.1) PROPOSITION. *Let Ω be a finite G-set and R a commutative ring such that the Krull-Schmidt theorem holds for RG-modules.*

1. *If Ξ is a graded set with Poincaré series f then*

$$P_{R\,\mathrm{Map}(\Omega,\Xi)} = \sum_{K \le J} \frac{R[G/K]\,\mu(K,J)\,f_J}{|G:K|}$$

in $A(G)[[t]]$, where the functions f_J are as given in 1.4.

2. *For each subgroup $J \le G$ define a function*

$$f_J = \frac{1}{(1-t^{|\Omega_1|})\cdots(1-t^{|\Omega_n|})}$$

where $\Omega = \Omega_1 \cup \cdots \Omega_n$ is the decomposition of Ω into J-orbits. Then the Poincaré series

$$P_{S^\bullet(R\Omega)} = \sum_{i=0}^{\infty} S^i(R\Omega)\,t^i$$

of the symmetric algebra on $R\Omega$ is equal to

$$\sum_{K \le J} \frac{R[G/K]\,\mu(K,J)\,f_J}{|G:K|}$$

in $A(G)[[t]]$.

PROOF. (1) This is immediate from 1.4 on replacing G-sets with permutation modules.

(2) is a particular instance of (1) in the case of the graded set $\Xi = \mathbb{N}$, which has Poincaré series $f(t) = \frac{1}{1-t}$. □

As with 1.4 there is a generalization where instead of considering polynomials in a set Ω in which every element has degree 1, we allow Ω itself to be a graded G-set so that some elements of Ω may be assigned different degrees. In this case a similar result to 2.1 holds, but in the denominator of f_J we replace $1 - t^{|\Omega_i|}$ by $1 - t^{d_i|\Omega_i|}$ where d_i is the degree of the elements in Ω_i.

We may be interested just in the ring of invariants $(S^\bullet(R\Omega))^G$ in $S^\bullet(R\Omega)$, not the full module structure. The theory in this case has been known for a long time and is described in [**9** , **Theorem 10.1**] (I am indepted to L.G. Kovács for this reference). The Poincaré series for this ring of invariants may also be computed using Molien's theorem, which ostensibly applies only in case $|G|$ is invertible in R, but may be applied here because of known properties of permutation modules. We wish to indicate also that the determination of the Poincaré series of the ring of invariants is a consequence of Proposition 2.1. Since each transitive permutation module $R[G/H]$ has fixed points of rank 1, we obtain the Poincaré series of the invariants from the series in 2.1 by replacing each module $R[G/H]$ by 1.

There is also a more direct way to obtain the Poincaré series of the invariants, which we take the opportunity to describe. We use Burnside's formula for the number of orbits of G on a set:

$$|G\backslash S^i(\Omega)| = \frac{1}{|G|}\sum_{g\in G} |(S^i(\Omega))^{\langle g\rangle}|.$$

(2.2) PROPOSITION. *The Poincaré series of the ring of invariants* $(S^\bullet(R\Omega))^G$ *is*

$$\frac{1}{|G|}\sum_{g\in G} f_{\langle g\rangle}(t)$$

where for each cyclic subgroup $J = \langle g\rangle$ *the functions* f_J *are as defined in 2.1.*

PROOF. Each transitive permutation module $R[G/H]$ has a fixed point set of R-rank 1, and so $\mathrm{rank}_R\, S^i(R\Omega)^G$ equals the number of orbits of G on $S^i(\Omega)$. Thus the Poincaré series is

$$\begin{aligned}\sum_{i=0}^\infty \mathrm{rank}_R(S^i(R\Omega)^G)t^i &= \sum_{i=0}^\infty |G\backslash S^i(\Omega)|t^i \\ &= \sum_{i=0}^\infty \frac{1}{|G|}\sum_{g\in G} |(S^i(\Omega))^{\langle g\rangle}|t^i \\ &= \frac{1}{|G|}\sum_{g\in G} f_{\langle g\rangle}(t)\end{aligned}$$

where as before

$$f_J(t) = \sum_{i=0}^\infty |(S^i(\Omega))^J|t^i.$$

□

Consider our example in which $G = C_2 \times C_2$ acts regularly on a set Ω of size 4. The Poincaré series of $S^\bullet(R\Omega)^G$ is

$$\begin{aligned}\frac{1}{4}(f_1 + f_A + f_B + f_C) &= \frac{1}{4}\left(\frac{1}{(1-t)^4} + \frac{3}{(1-t^2)^2}\right) \\ &= \frac{1-t+t^2}{(1-t)^2(1-t^2)^2} \\ &= 1 + t + 4t^2 + 5t^3 + 11t^4 + 14t^5 + 24t^6 + 30t^7 + \cdots\end{aligned}$$

We note that the approach we have just described is merely a formalisation of a method used in [**1**].

3. The cohomology of wreath products

We now apply our results on permutation modules to give the following result in cohomology.

(3.1) THEOREM. *Let $X = F \wr G$ be a wreath product of finite groups, where G permutes a product of copies of F indexed by a G-set Ω. Let*

$$f(t) = \sum \dim H^n(F, \mathbb{F}_p) t^n$$

be the Poincaré series of the cohomology ring of F, and for each subgroup $J \leq G$ let $f_J(t) = f(t^{|\Omega_1|}) \cdots f(t^{|\Omega_n|})$ where $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$ is the decomposition of Ω into J-orbits. Also let $g_J(t)$ be the Poincaré series of the cohomology ring of J. Then the Poincaré series of the cohomology ring of X is

$$\Phi(t) = \sum_{K \leq J} \frac{g_K \, \mu(K, J) \, f_J}{|G : K|}$$

where the sum ranges over all pairs of subgroups $K \leq J$ of G.

Thus $\Phi(t)$ is obtained by substituting g_K for G/K in the series $P_{\mathrm{Map}(\Omega,\Xi)}(t)$ where Ξ is any graded set with Poincaré series $f(t)$ (for example, a basis of $H^*(F, \mathbb{F}_p)$).

PROOF. Our starting point is the isomorphism

$$H^*(F \wr G, \mathbb{F}_p) \cong H^*(G, H^*(F, \mathbb{F}_p)^{(m)})$$

shown by Nakaoka [**8** , **p.237**] using work of Steenrod and quoted by Bogačenko [**3**]. Here $H^*(F, \mathbb{F}_p)^{(m)}$ is the m-fold tensor product, where $m = |\Omega|$, and G acts on it by permuting the factors. Thus it is immediate that if Ξ is a basis of $H^*(F, \mathbb{F}_p)$, taken as a graded set so that $\Xi(n)$ is a basis of $H^n(F, \mathbb{F}_p)$ for each n, then the basis elements $\xi_1 \otimes \cdots \otimes \xi_m$, $\xi_i \in \Xi$, $i \in \Omega$ of the tensor power are in bijection with functions $\Omega \to \Xi$. The action of G on $H^*(F, \mathbb{F}_p)^{(m)}$ arises from the permutation action of G on this basis, which under the correspondence with $\mathrm{Map}(\Omega, \Xi)$ is the same as the action on $\mathrm{Map}(\Omega, \Xi)$ defined in Section 1, since it arises from the action on Ω. It follows that $H^*(F, \mathbb{F}_p)^{(m)} \cong \mathbb{F}_p \mathrm{Map}(\Omega, \Xi)$ as $\mathbb{F}_p G$-modules.

The Poincaré series of Ξ is the same as the Poincaré series of $H^*(F, \mathbb{F}_p)$, namely f. Thus the series in $A(G)[[t]]$ of the graded $\mathbb{F}_p G$-module $H^*(F, \mathbb{F}_p)^{(m)}$ is

$$\sum_{K \leq J} \frac{\mathbb{F}_p[G/K] \, \mu(K, J) \, f_J}{|G : K|}$$

by 2.1. The theorem now follows from the next lemma.

(3.2) LEMMA. *Suppose M is a graded permutation module over $\mathbb{F}_p G$ with series*

$$\sum_{i=0}^{\infty} M(i) t^i = \sum_{K \leq G} \mathbb{F}_p[G/K] p_K(t) \in A(G)[[t]]$$

where $p_K(t) \in \mathbb{Z}[[t]]$. Then $H^*(G, M)$ has series

$$\sum_{K \leq G} g_K(t) p_K(t)$$

where g_K is the Poincaré series of $H^*(K, \mathbb{F}_p)$.

PROOF. On taking cohomology, each of the modules $\mathbb{F}_p[G/K]$ gives rise to a term $H^*(G, \mathbb{F}_p[G/K]) \cong H^*(K, \mathbb{F}_p)$ and so contributes $g_K(t)$ to the Poincaré series of M. □

(3.3) EXAMPLE. The mod p cohomology of $F \wr C_p$. Let $G = C_p$ acting regularly on a set Ω of size p, and let F be any group with Poincaré series f. We have

$$g_1(t) = 1 \qquad f_1(t) = f(t)^p$$

$$g_G(t) = \tfrac{1}{1-t} \quad f_G(t) = f(t^p)$$

and

$$\Phi = g_G f_G - \frac{g_1 f_G}{p} + \frac{g_1 f_1}{p}$$

which is the formula obtained by Bogačenko [**3**].

(3.4) EXAMPLE. The mod 2 cohomology of $F \wr (C_2 \times C_2)$. Here $G = C_2 \times C_2$ acting regularly on a set Ω of size 4. As usual, we denote the three subgroups of G of order 2 by A, B and C, and we have

$$g_1(t) = 1 \qquad f_1(t) = f(t)^4$$

$$g_A(t) = g_B(t) = g_C(t) = \tfrac{1}{1-t} \quad f_A(t) = f_B(t) = f_C(t) = f(t^2)^2$$

$$g_G(t) = \tfrac{1}{(1-t)^2} \qquad f_G = f(t^4).$$

Substituting this into formula 1.3 gives

$$\begin{aligned}\Phi(t) =& f(t^4)\left(\frac{1}{(1-t)^2} - \frac{3}{2(1-t)} + \frac{1}{4}\right)\\ &+ 3f(t^2)^2\left(\frac{1}{2(1-t)} - \frac{1}{4}\right)\\ &+ \frac{f(t)^4}{4}\\ =& f(t^4)\frac{-1+4t+t^2}{4(1-t)^2} + f(t^2)^2\frac{3(1+t)}{4(1-t)} + \frac{f(t)^4}{4}.\end{aligned}$$

As a particular example we consider the cohomology of $C_2 \wr (C_2 \times C_2)$, which is of interest since it is the Sylow 2-subgroup of $GL(4,2) \cong A_8$. One obtains this identification of the Sylow 2-subgroup of these groups on noting that the Sylow 2-subgroup of $GL(4,2)$ is a semidirect product

$$\begin{pmatrix}1 & 0 & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \rtimes \begin{pmatrix}1 & * & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & *\\ 0 & 0 & 0 & 1\end{pmatrix}$$

where the conjugation action of the quotient $C_2 \times C_2$ is as the regular representation on the subgroup $(C_2)^4$. For the group C_2 the Poincaré series is $f(t) = \frac{1}{1-t}$ and so the Poincaré series of the cohomology of $C_2 \wr (C_2 \times C_2)$ is

$$\begin{aligned}\Phi(t) &= \frac{1+t^2-t^3}{(1-t)^3(1-t^4)}\\ &= 1 + 3t + 7t^2 + 12t^3 + 19t^4 + 28t^5 + 40t^6 + 54t^7 + 71t^8 + 91t^9 + \cdots .\end{aligned}$$

This cohomology has been computed in [**10**].

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