RESOLUTIONS, RELATION MODULES AND SCHUR MULTIPLIERS FOR CATEGORIES

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Abstract. We show that the construction in group cohomology of the Gruenberg resolution associated to a free presentation and the resulting relation module can be copied in the context of representations of categories. We establish five-term exact sequences in the cohomology of categories and go on to show that the Schur multiplier of the category has properties which generalize those of the Schur multiplier of a group.

To the memory of Karl Gruenberg.

1. Introduction

It has been known for some time that many results from the representation theory and cohomology of groups may be made to work in the more general setting of representations and cohomology of categories. A brief exposition of some of this theory is given in [19] where it is explained that the usual interpretation of low dimensional group cohomology works in the setting of cohomology of categories, including the parametrization of equivalence classes of extensions by elements of second cohomology, and a familiar interpretation of first cohomology in terms of complements in the Grothendieck construction (which generalizes the semidirect product of groups).

In this paper we turn to the Schur multiplier, defined for categories in the same way as for groups, as the second homology group. We show that the usual theory for groups generalizes to categories, when suitably formulated. This theory has to do with central extensions, and two key results in group theory are, firstly, that in any central extension where the kernel is contained in the derived subgroup, the kernel is an image of the Schur multiplier; and secondly that there

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is up to isomorphism a unique maximal perfect central extension of a perfect group. We show that these and other results do work in the larger context of categories, but to do this we must first say what we mean by a central extension of a category, and also formulate a condition which generalizes the notion that the kernel of an extension is contained in the derived subgroup. There are in fact several ways to generalize the first of these conditions, and we discuss them here. We also show that the 5-term sequences from group cohomology generalize to categories. These are a useful tool in dealing with Schur multipliers.

As well as generalizing the theory of central extensions, we introduce a theory of relation modules for categories which extends the notion from group theory [10]. The initial problem is to say what we might even mean by a relation module, since the usual group theoretic construction is not available. In group theory a relation module is the abelianized quotient of the relation subgroup in a free presentation of the group. With categories, we may take a presentation of a category by a free category (a surjective functor between the categories, the notion of a *free category* being defined in [17, II.7]) but there is no adequate analogue of the relation subgroup whose abelianization we hope to take.

This is an example of a general obstacle, which is that techniques from group theory which involve computation within groups are frequently not available when working with categories. On the other hand techniques which use representations often do translate more easily, and we emphasize such methods. To define a relation module we first show that starting from a surjection from a free category to another given category, there is a projective resolution of the constant functor which is analogous to the Gruenberg resolution in group theory. The relation module is defined as the second kernel in this resolution. This is a new construction, but we show that in case the category we start with is a group we obtain either the usual relation module or else something very closely related to it. We also show that the relation module is the kernel in a projective object in a certain category whose objects are category extensions.

These categories whose objects are extensions provide a way to describe the fundamental technique which we shall use. Gruenberg [9] showed that the category whose objects are group extensions with abelian kernel is equivalent to the category of module extensions of the augmentation ideal of the group. It turns out that a generalized version of this result is true for category extensions also. The functors which give this equivalence are the tools which allow us to translate between categories on the one hand, and representations of categories on the other.

We will use the same notation as in [19], to which we also refer for basic background on the representation theory and cohomology of categories, such as facts about the constant functor, left and right augmentation ideals, and so on. A discussion of background and the uses of representations of categories can also be found in many of the other references we give, including [1, 2, 3, 6, 8, 11, 14, 15, 18]. In any case, we now give some of the basic definitions. For convenience we will work with small categories with finitely many objects, although the theory can be extended more generally than this. A *representation* of such a category $\mathcal{C}$ over a commutative ring R with identity is a functor $M : \mathcal{C} \to R$-mod. We will adopt the convention that we compose morphisms on the left, so that $\beta\alpha$ means first do α then β. The *category algebra* $R\mathcal{C}$ is the free R-module with the morphisms of $\mathcal{C}$ as a basis, the multiplication being composition of morphisms when defined, and otherwise zero. It has the property that representations of $\mathcal{C}$ over R may be regarded as left $R\mathcal{C}$-modules, and vice-versa. Equally, right $R\mathcal{C}$-modules, which are the same thing as $R\mathcal{C}^{\mathrm{op}}$-modules, may be identified with functors $\mathcal{C}^{\mathrm{op}} \to R$-mod, where $\mathcal{C}^{\mathrm{op}}$ is the opposite category. There is a *left constant functor* $\underline{R}$ which is the representation of $\mathcal{C}$ which assigns to each object the R-module R, and to each morphism the identity map. There is also a *right constant functor*, which is the left constant functor for $\mathcal{C}^{\mathrm{op}}$. The category algebra has two augmentation ideals, the *left augmentation ideal* ${}^{\bullet}I\mathcal{C}$ and the *right augmentation ideal* $I\mathcal{C}^{\bullet}$. The left augmentation is the kernel of the map of left $R\mathcal{C}$-modules $R\mathcal{C} \to \underline{R}$ which sends each morphism $\alpha : x \to y$ in $\mathcal{C}$ to 1 in the copy of R corresponding to y. Thus ${}^{\bullet}I\mathcal{C}$ is the R-submodule of $R\mathcal{C}$ spanned by the elements $1_y - \alpha$ where $y = \mathrm{cod}(\alpha)$ is the codomain of α. The right augmentation is the kernel of the map of left $R\mathcal{C}$-modules $R\mathcal{C} \to \underline{R}$ which sends each morphism $\alpha : x \to y$ in $\mathcal{C}$ to 1 in the copy of R corresponding to x. Thus ${}^{\bullet}I\mathcal{C}$ is the R-submodule of $R\mathcal{C}$ spanned by the elements $1_x - \alpha$ where $x = \mathrm{dom}(\alpha)$ is the domain of α.

A number of times we will use the *Grothendieck construction*, in the special case when we are given a representation M of $\mathcal{C}$. This is a category $M \rtimes \mathcal{C}$ which we may take to have the same objects as $\mathcal{C}$ and where the morphisms $x \to y$ are pairs (m, α) with $\alpha : x \to y$ in $\mathcal{C}$ and $m \in M(y)$. Given another morphism $(n, \beta) : y \to z$ the composition is defined to be $(n, \beta) \circ (m, \alpha) = (n + {}^{\beta}m, \beta\alpha)$ with ${}^{\beta}m$ denoting the functorial action of β on m.

In defining the homology and cohomology of a small category $\mathcal{C}$ we assume that A is a left $R\mathcal{C}$-module, B is a right $R\mathcal{C}$-module and take

$H_*(\mathcal{C}, B) = \mathrm{Tor}_*^{R\mathcal{C}}(B, \underline{R})$ and $H^*(\mathcal{C}, A) = \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, A)$ where $\underline{R}$ is the constant functor. When dealing with group cohomology, for example, it does not make much difference if B is a right module or a left module, but here the distinction is significant, and our convention with H_* and right modules is in keeping with [12], for example.

Here is the layout of this paper: we start in Section 2 by showing that the Gruenberg resolution from group cohomology may be made to work equally well with categories, giving formulas for the homology and cohomology with constant functor coefficients. The construction of the start of that resolution is fundamental to the rest of what we do in this paper. It is used immediately in Section 3 where we show that extensions of a category may be translated into extensions of the left augmentation ideal, and vice-versa. In the special case when we start with a presentation of a category by a free category (i.e. a surjection from a free category) the construction produces what we define to be a relation module, and this is described in Section 4. We next turn to the properties of the Schur multiplier, defined for a category $\mathcal{C}$ as $H_2(\mathcal{C}, \underline{\mathbb{Z}})$. We first develop a useful tool for establishing these properties, namely five-term exact sequences which are analogous to such sequences in group cohomology. We show in Section 5 that there are two kinds of such sequences, one kind associated to an extension of categories, the other associated to the opposite categories in an extension. Continuing with the discussion of the Schur multiplier, we formulate the analogues for category extensions of the notion of central extension, and stem extension. This is begun in Section 6 where we also deduce immediate consequences for the Schur multiplier of the five-term sequence and the universal coefficient theorem. The most obvious generalization to categories of the notion of a central extension of groups is to consider extensions by constant functors, or at least locally constant functors (which are constant on connected components of the category $\mathcal{C}$ being represented). It turns out that the arguments we wish to make require a still more general notion, namely extensions by sublocally constant functors, which we define to be subfunctors of locally constant functors. We develop their properties in Section 7, and move on in Section 8 to describe the maximal stem extensions, and to prove that when $\mathcal{C}$ is a category with free abelian first homology and finitely generated second homology, there is a unique isomorphism class of maximal stem extensions by locally constant functors. In Section 9 we conclude with some examples.

It will be apparent to anyone familiar with Karl Gruenberg's book [9] that much of what I have done here is to go through his work and present his arguments in the context of categories. The influence of Gruenberg's development of the theory and of his teaching is pervasive, and I acknowledge with gratitude the debt which I owe.

2. A Gruenberg Resolution for Categories

We start with the construction which will be fundamental in the next sections. Let $\mathcal{C}$ and $\mathcal{E}$ be categories which have the same finite set of objects, and let $p : \mathcal{E} \to \mathcal{C}$ be a functor which is surjective on morphisms. Such a functor extends to an algebra homomorphism $R\mathcal{E} \to R\mathcal{C}$ whose kernel we will denote N, a 2-sided ideal in $R\mathcal{E}$. Taking N and the left augmentation ideal ${}^{\bullet}I\mathcal{E}$ we will construct an acyclic complex of $R\mathcal{C}$-modules in the same way as the Gruenberg resolution is constructed in [9, p. 34], [7, Sec. 1, Prop. 3] and [4, Prop. 2.4]. Subsequently our goal in this section will be to show that when $\mathcal{E}$ is a free category this complex is a resolution of $\underline{R}$ by projective $R\mathcal{C}$-modules after which we will use the resolution to give formulas for the homology and cohomology of $\mathcal{C}$.

First we record some facts about the ideals N and ${}^{\bullet}I\mathcal{E}$. The first (which is well known) is an extension of the fact that over a field the category algebra of a free category is hereditary.

Proposition 2.1. *Let $\mathcal{F}$ be a free category and R a commutative ring with a 1. Left ideals $I \subseteq R\mathcal{F}$ for which $R\mathcal{F}/I$ is projective as an R-module are projective as $R\mathcal{F}$-modules.*

Proof. The argument we use is based on one which appears in [4], the difference here being that we allow an arbitrary ground ring instead of a field. Let $J \subset R\mathcal{F}$ be the ideal which is the span of the non-identity morphisms of $\mathcal{F}$. We have an exact sequence of $(R\mathcal{F}, R\mathcal{F})$-bimodules

$$0 \to R\mathcal{F} \otimes_R J \otimes_R R\mathcal{F} \to R\mathcal{F} \otimes_R R\mathcal{F} \to R\mathcal{F} \to 0$$

as in [4] where the left non-zero map is $u \otimes \alpha \otimes v \mapsto u\alpha \otimes v - u \otimes \alpha v$ and the right non-zero map is $\alpha \otimes \beta \mapsto \alpha\beta$. When regarded as a sequence of right $R\mathcal{F}$-modules this sequence is split since the right hand term is projective. Thus if we write $M = R\mathcal{F}/I$ and apply $- \otimes_{R\mathcal{F}} M$ to it, using the identification $R\mathcal{F} \otimes_{R\mathcal{F}} M \cong M$ and associativity of tensor products, we obtain a sequence

$$0 \to R\mathcal{F} \otimes_R J \otimes_R M \to R\mathcal{F} \otimes_R M \to M \to 0$$

which is exact because the previous sequence of right modules was split. Since M is projective as an R-module, as is J, the two modules on the left hand side of this sequence are projective as left $R\mathcal{F}$-modules. Applying Schanuel's lemma to this sequence and the short exact sequence of left $R\mathcal{F}$-modules

$$0 \to I \to R\mathcal{F} \to M \to 0$$

we deduce that

$$I \oplus (R\mathcal{F} \otimes_R M) \cong R\mathcal{F} \oplus (R\mathcal{F} \otimes_R J \otimes_R M),$$

from which it follows that I is projective. □

We deduce from this that when $\mathcal{F}$ is a free category the ideals N and ${}^{\bullet}I\mathcal{F}$ are projective $R\mathcal{F}$-modules since the factor modules $R\mathcal{F}/N \cong R\mathcal{C}$ and $R\mathcal{F}/{}^{\bullet}I\mathcal{F} \cong \underline{R}$ are both projective as R-modules. In the case of the left augmentation ideal ${}^{\bullet}I\mathcal{F}$ we can do better than this. We will use the notation $F_x = R\operatorname{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to R\text{-mod}$ for the linearized representable functor associated to the object x of $\mathcal{C}$; thus $F_x(y) = R\operatorname{Hom}_{\mathcal{C}}(x, y)$ is the free R-module with the homomorphisms from x to y as a basis. The $R\mathcal{C}$-module F_x is projective by Yoneda's lemma [19, Prop. 4.4]. We comment that some authors refer to direct sums of the F_x as free representations, but we prefer to reserve this terminology for representations isomorphic to a direct sum of copies of $R\mathcal{C}$.

Proposition 2.2. *Let $\mathcal{F}$ be a free category with morphisms $\alpha_1, \alpha_2, \dots, \alpha_n$ as free generators. Then*

$$\begin{aligned} {}^{\bullet}I\mathcal{F} &= R\mathcal{F}(1_{\operatorname{cod}(\alpha_1)} - \alpha_1) \oplus \cdots \oplus R\mathcal{F}(1_{\operatorname{cod}(\alpha_n)} - \alpha_n) \\ &\cong F_{\operatorname{cod}(\alpha_1)} \oplus \cdots \oplus F_{\operatorname{cod}(\alpha_n)} \end{aligned}$$

as left $R\mathcal{F}$-modules, where $\operatorname{cod}(\alpha)$ denotes the codomain of a morphism α.

Proof. A direct argument with elements of the augmentation ideal is possible. We present here an indirect argument which copies the corresponding argument for groups which is used to prove [9, Sect. 3.1, Prop. 1]. We sketch the approach, and the reader can refer to [9] for greater detail.

We start with the isomorphism

$$\operatorname{Hom}_{R\mathcal{F}}({}^{\bullet}I\mathcal{F}, M) \cong \operatorname{Der}(\mathcal{F}, M)$$

for any representation M under which a homomorphism $\delta : {}^{\bullet}I\mathcal{F} \to M$ corresponds to a derivation $d : \mathcal{F} \to M$ with $d(\alpha) = \delta(1_{\operatorname{cod}(\alpha)} - \alpha)$ (see [19, Lemma 6.2] for this result and the definition of a derivation). Next we form the Grothendieck construction $M \rtimes \mathcal{F}$ and observe that a mapping $s : \mathcal{F} \to M \rtimes \mathcal{F}$ of the form $\alpha \mapsto (d(\alpha), \alpha)$ is a functor if and only if $d : \mathcal{F} \to M$ is a derivation. Because $\mathcal{F}$ is free any assignment of the generators $d(\alpha_i) \in M(\operatorname{cod}(\alpha_i))$ gives rise to a unique functor s with $s(\alpha_i) = (d(\alpha_i), \alpha_i)$ for all i, and so extends to a unique derivation

$d : \mathcal{F} \to M$. Thus any assignment of the elements $1_{\mathrm{cod}(\alpha_i)} - \alpha_i$ to elements in $M(\mathrm{cod}(\alpha_i))$ extends uniquely to a homomorphism of $R\mathcal{F}$-modules ${}^\bullet I\mathcal{F} \to M$. This property characterizes ${}^\bullet I\mathcal{F}$ as the projective module indicated. $\square$

Although the same in spirit as resolutions construction by Gruenberg [9] and Butler-King [4], the projective resolution specified in the next theorem, and the ensuing corollaries for homology, do not appear to be consequences of that earlier work because the setting we consider is different. Butler and King consider presentations of algebras by tensor algebras (category algebras of free categories) where the kernel is contained in the square of the ideal spanned by the generators of the algebra. This restriction plays a significant role in their work, and it is important that we do not make it here, so that for example, presentations of groups by free monoids may be considered.

Theorem 2.3. *Let $\mathcal{E}$ and $\mathcal{C}$ be categories with the same objects, let $p : \mathcal{E} \to \mathcal{C}$ be a surjection of categories which is the identity on objects and let $N = \mathrm{Ker}(R\mathcal{E} \to R\mathcal{C})$. Consider the acyclic complex of $R\mathcal{C}$-modules*

$$\cdots \to \frac{N^t}{N^{t+1}} \to \frac{N^{t-1}\cdot {}^\bullet I\mathcal{E}}{N^t\cdot {}^\bullet I\mathcal{E}} \to \frac{N^{t-1}}{N^t} \to \cdots$$
$$\to \frac{N}{N^2} \to \frac{{}^\bullet I\mathcal{E}}{N\cdot {}^\bullet I\mathcal{E}} \to R\mathcal{C} \to \underline{R} \to 0.$$

If $\mathcal{E}$ is a free category this complex is an $R\mathcal{C}$-projective resolution of $\underline{R}$.

Proof. The acyclic complex arises from the chain of left ideals

$$\cdots \subseteq N^3 \subseteq N^2\cdot {}^\bullet I\mathcal{E} \subseteq N^2 \subseteq N\cdot {}^\bullet I\mathcal{E} \subseteq N \subseteq {}^\bullet I\mathcal{E} \subseteq R\mathcal{E}$$

which form the list of numerators in the complex, and also the list of denominators when shifted by 2. The inclusion maps from each ideal to the next induce the maps in the complex.

When $\mathcal{E}$ is free, by Proposition 2.1 both N and ${}^\bullet I\mathcal{E}$ are projective $R\mathcal{E}$-modules, and hence all products N^n and $N^n\cdot {}^\bullet I\mathcal{E}$ are projective left $R\mathcal{E}$-modules by Remark 5 in [9, p. 35], attributed there to Kaplansky. Since the modules in the resolution (apart from $\underline{R}$) are obtained from these products by applying $R\mathcal{C}\otimes_{R\mathcal{E}} -$, they are projective $R\mathcal{C}$-modules. $\square$

Corollary 2.4. *Let $\mathcal{F}$ and $\mathcal{C}$ be categories with the same objects, let $p : \mathcal{F} \to \mathcal{C}$ be a surjection of categories which is the identity on objects, and let $N = \mathrm{Ker}(R\mathcal{F} \to R\mathcal{C})$. Suppose that $\mathcal{F}$ is a free category. Then*

$$H_{2n}(\mathcal{C}, \underline{R}) \cong \frac{N^n \cap I\mathcal{F}^\bullet \cdot N^{n-1}\cdot {}^\bullet I\mathcal{F}}{I\mathcal{F}^\bullet\cdot N^n + N^n\cdot {}^\bullet I\mathcal{F}}$$

and

$$H_{2n+1}(\mathcal{C},\underline{R}) \cong \frac{I\mathcal{F}^\bullet \cdot N^n \cap N^n \cdot {}^\bullet I\mathcal{F}}{N^{n+1} + I\mathcal{F}^\bullet \cdot N^n \cdot {}^\bullet I\mathcal{F}}.$$

The formula for first homology simplifies to give

$$H_1(\mathcal{C},\underline{R}) \cong \frac{I\mathcal{C}^\bullet \cap {}^\bullet I\mathcal{C}}{I\mathcal{C}^\bullet \cdot {}^\bullet I\mathcal{C}}.$$

Proof. We use the monomorphisms

$$0 \to \frac{N^n}{N^n \cdot {}^\bullet I\mathcal{F}} \to \frac{N^{n-1} \cdot {}^\bullet I\mathcal{F}}{N^n \cdot {}^\bullet I\mathcal{F}}$$

and

$$0 \to \frac{N^n \cdot {}^\bullet I\mathcal{F}}{N^{n+1}} \to \frac{N^n}{N^{n+1}}$$

which come from the resolution to compute homology. We apply the functor $\underline{R} \otimes_{R\mathcal{C}} -$, which on any $R\mathcal{C}$-module M can be identified as taking the value $M/(I\mathcal{F}^\bullet \cdot M)$. We get exact sequences

$$0 \to H_{2n}(\mathcal{C},\underline{R}) \to \frac{N^n}{I\mathcal{F}^\bullet \cdot N^n + N^n \cdot {}^\bullet I\mathcal{F}} \to \frac{N^{n-1} \cdot {}^\bullet I\mathcal{F}}{I\mathcal{F}^\bullet \cdot N^{n-1} \cdot {}^\bullet I\mathcal{F}}$$

and

$$0 \to H_{2n+1}(\mathcal{C},\underline{R}) \to \frac{N^n \cdot {}^\bullet I\mathcal{F}}{N^{n+1} + I\mathcal{F}^\bullet \cdot N^n \cdot {}^\bullet I\mathcal{F}} \to \frac{N^n}{I\mathcal{F}^\bullet \cdot N^n}.$$

From these the homology groups are identified as stated in general.

When $n = 0$ we get

$$H_1(\mathcal{C},\underline{R}) \cong \frac{I\mathcal{F}^\bullet \cap {}^\bullet I\mathcal{F}}{N + I\mathcal{F}^\bullet \cdot {}^\bullet I\mathcal{F}}.$$

We have identifications ${}^\bullet I\mathcal{C} \cong {}^\bullet I\mathcal{F}/N$ and $I\mathcal{C}^\bullet \cong I\mathcal{F}^\bullet/N$ so that

$$I\mathcal{C}^\bullet \cap {}^\bullet I\mathcal{C} \cong (I\mathcal{F}^\bullet \cap {}^\bullet I\mathcal{F})/N$$

and

$$I\mathcal{C}^\bullet \cdot {}^\bullet I\mathcal{C} \cong (N + I\mathcal{F}^\bullet \cdot {}^\bullet I\mathcal{F})/N.$$

From these the formula for $H_1(\mathcal{C},\underline{R})$ follows using the third isomorphism theorem. □

These formulas fit into a picture exactly exactly like that on [9, p. 48]. The formula for first homology extends the result for groups that the abelianization of the group is isomorphic to the augmentation ideal (over the integers) modulo its square.

At this point we record also the universal coefficients formula, which allows us to deduce the cohomology of a category with constant coefficients from the formulas for homology in 2.4. It is an instance of standard theory.

Theorem 2.5 (Universal Coefficient Theorem). *Let A be an abelian group. There are short exact sequences which are functorial in A and which split, but not naturally:*

$$0 \to H_n(\mathcal{C}, \underline{\mathbb{Z}}) \otimes A \to H_n(\mathcal{C}, \underline{A}) \to \mathrm{Tor}(H_{n-1}(\mathcal{C}, \underline{\mathbb{Z}}), A) \to 0$$

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}, \underline{\mathbb{Z}}), A) \to H^n(\mathcal{C}, \underline{A}) \to \mathrm{Hom}(H_n(\mathcal{C}, \underline{\mathbb{Z}}), A) \to 0.$$

Proof. See [12]. □

3. An Equivalence of Extension Categories

In this section we describe two categories whose objects are extensions, in the first case extensions of a fixed category $\mathcal{C}$ and in the second case $R\mathcal{C}$-module extensions of ${}^{\bullet}I\mathcal{C}$. The extensions of $\mathcal{C}$ as a category are what we are really interested in, but it is technically easier to work with the category of module extensions of ${}^{\bullet}I\mathcal{C}$. We construct functors in both directions between the two extension categories which allow us to translate questions from categories to modules, and back again. Our main goal is Theorem 3.6 which states that the two extension categories we are about to define are equivalent. After that, we show in Theorem 3.7 that the same ideas also work in a more general setting.

We first recall the notion of an extension of a category which appears in Hoff [13], bearing in mind that there are also other approaches to category extensions (see [1, 8, 11, 13] and also [19] for an exposition of this basic material). An extension of categories is a pair of functors

$$\mathcal{K} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}$$

between categories $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$ for which

(1) $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$ all have the same objects, i and p are the identity on objects, i is injective on morphisms, and p is surjective on morphisms;
(2) whenever f and g are morphisms in $\mathcal{E}$ then $p(f) = p(g)$ if and only if there exists a morphism $m \in \mathcal{K}$ for which $f = i(m)g$. In that case, the morphism m is required to be unique.

It is convenient to use the notation $(\mathcal{K}|\mathcal{E})$ to denote the above extension, although this notation does not retain complete information about it. We leave the reader to prove it as a exercise or consult the above references to see that $\mathcal{K}$ must be a disjoint union of groups.

The category $\mathcal{K}$ which appears in the extension has further structure, and may in fact be regarded a functor K from $\mathcal{E}$ to Groups, generalizing the fact that the kernel in a group extension acquires an action of the extension group coming from conjugation. Thus if $\alpha : x \to y$ is a morphism in $\mathcal{E}$ and $m \in \mathrm{End}_{\mathcal{K}}(x)$ then $p(\alpha) = p(\alpha i(m))$, and so $\alpha i(x) = i({}^{\alpha}m)\alpha$ for some unique element ${}^{\alpha}m \in \mathrm{End}_{\mathcal{K}}(y)$. The specification $K(x) = \mathrm{End}_{\mathcal{K}}(x)$ and $K(\alpha)(m) = {}^{\alpha}m$ defines a functor K constructed from the extension. If the target groups $K(x)$ of this functor happen to be abelian then K factors through the surjective functor $p : \mathcal{E} \to \mathcal{C}$ as a composite $\mathcal{E} \xrightarrow{p} \mathcal{C} \to$ Abelian Groups and we write K also for the functor $\mathcal{C} \to$ Abelian Groups so obtained. In this situation the category $\mathcal{K}$ may thus be regarded as a $\mathbb{Z}\mathcal{C}$-module K. The ambiguity between regarding the left term in the extension as a category $\mathcal{K}$ or as a $\mathbb{Z}\mathcal{C}$-module K will be present throughout the remainder of this work, and we use both symbols $\mathcal{K}$ and K for it.

In any case, when the $K(x)$ are not abelian the homomorphisms $K(\alpha) : K(x) \to K(y)$ pass to homomorphisms

$$K(x)/K(x)' \to K(y)/K(y)'$$

of the abelianized groups, and on these $\mathcal{K}$ acts trivially. The first homology $H_1(\mathcal{K})$ is the product of these abelianizations (since $\mathcal{K}$ is a disjoint union of groups) and so $H_1(\mathcal{K})$ acquires the structure of a left $\mathbb{Z}\mathcal{C}$-module. This module structure will appear throughout the rest of this paper.

There is a bijection between the equivalence classes of representations of $\mathcal{C}$ by a representation K and the elements of $H^2(\mathcal{C}, K)$, under which the zero element corresponds to the split extension $K \to K \rtimes \mathcal{C} \to \mathcal{C}$ where $K \rtimes \mathcal{C}$ is the Grothendieck construction [19, 7.3]. Thus to any extension of $\mathcal{C}$ by K we may associate a second cohomology class $\zeta \in H^2(\mathcal{C}, K)$, and if $f : K \to K_1$ is any homomorphism of $R\mathcal{C}$-modules we obtain by functoriality a second cohomology class $f_*(\zeta) \in H^2(\mathcal{C}, K_1)$. We point out that given any commutative diagram of functors where the rows are extensions

$$\begin{array}{cccccc}
\zeta : & K & \xrightarrow{i} & \mathcal{E} & \longrightarrow & \mathcal{C} \\
& \downarrow f & & \downarrow & & \| \\
& K_1 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{C}
\end{array}$$

the class of the lower extension is necessarily $f_*(\zeta)$, as may be seen by noting that ζ and $f_*(\zeta)$ are the images in cohomology of homomorphisms from the second kernel in a projective resolution of $\underline{\mathbb{Z}}$. Given f and ζ we may always construct an extension whose class is $f_*(\zeta)$ in the same way as for group extensions, as follows. We may define $\mathcal{E}_1$ to be the quotient of $K_1 \rtimes \mathcal{E}$ by $H = \{(-f(a), i(a)) \mid a \in K\}$. This means we form the category with the same objects as $\mathcal{C}$ and define $\mathrm{Hom}_{\mathcal{E}_1}(x, y) = H \backslash \mathrm{Hom}_{K_1 \rtimes \mathcal{E}}(x, y)$ as the set of orbits in the left action of H. The functors in the lower extension are induced by the inclusion into and projection from the Grothendieck construction $K_1 \rtimes \mathcal{E}$. We choose to call the lower extension in this construction the *explicit pushout* of the top extension along f.

We introduce two extension categories $\left(\frac{\mathcal{C}}{\ }\right)$ and $\left(\frac{{}^\bullet I\mathcal{C}}{\ }\right)$ as in [9, Chapter 9]. The objects of $\left(\frac{\mathcal{C}}{\ }\right)$ are the category extensions $(K|\mathcal{E}) = K \to \mathcal{E} \to \mathcal{C}$ where K may be any $R\mathcal{C}$-module, and the morphisms are the pairs of functors (f, g) so that

$$\begin{array}{ccccc} K & \to & \mathcal{E} & \to & \mathcal{C} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \| \\ K_1 & \to & \mathcal{E}_1 & \to & \mathcal{C} \end{array}$$

commutes. Similarly $\left(\frac{{}^\bullet I\mathcal{C}}{\ }\right)$ has as its objects the $R\mathcal{C}$-module extensions $(K|E) = K \to E \to {}^\bullet I\mathcal{C}$ and as morphisms the pairs of $R\mathcal{C}$-module homomorphisms (f, g) so that

$$\begin{array}{ccccc} K & \to & E & \to & {}^\bullet I\mathcal{C} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \| \\ K_1 & \to & E_1 & \to & {}^\bullet I\mathcal{C} \end{array}$$

commutes.

Certain formalities to do with these categories follow immediately as in [9] and we state some of them now.

Proposition 3.1. (1) *Let (f, g) be a morphism in $\left(\frac{\mathcal{C}}{\ }\right)$ or in $\left(\frac{{}^\bullet I\mathcal{C}}{\ }\right)$. The functor f is necessarily a homomorphism of $R\mathcal{C}$-modules. The morphism (f, g) is an epimorphism if and only if f is surjective, and is an isomorphism if and only if f is an isomorphism.*

(2) *Let $(K|\mathcal{E})$ and $(K_1|\mathcal{E}_1)$ be extensions of the category $\mathcal{C}$ with cohomology classes $\zeta \in H^2(\mathcal{C}, K)$ and $\eta \in H^2(\mathcal{C}, K_1)$. Suppose $f : K \to K_1$ is an $R\mathcal{C}$-module homomorphism for which $f_*(\zeta) = \eta$. Then there is a morphism of extensions of the form*

$$\begin{array}{ccccc} K & \to & \mathcal{E} & \to & \mathcal{C} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow & & \| \\ K_1 & \to & \mathcal{E}_1 & \to & \mathcal{C} \end{array}$$

A similar statement holds for the module extensions of ${}^\bullet I\mathcal{C}$.

Proof. These assertions are analogous to statements in [9] and are proved in the same way. We leave (1) to the reader, and (2) is really the 'Surjectivity Theorem' on page 187 of [9]. To prove it, we observe that the extension constructed from $(K|\mathcal{E})$ by explicit pushout along f has class $f_*(\zeta)$, and appears as the lower row of a diagram

$$\begin{array}{ccccc} K & \to & \mathcal{E} & \to & \mathcal{C} \\ \big\downarrow f & & \big\downarrow & & \| \\ K_1 & \to & \mathcal{E}_2 & \to & \mathcal{C} \end{array}.$$

This extension is equivalent to $(K_1|\mathcal{E}_1)$ and so composing the above diagram with the equivalence gives a morphism $(K|\mathcal{E}) \to (K_1|\mathcal{E}_1)$ as required. The argument in the category of module extensions is similar. □

We now construct functors between these extension categories. The first is a functor $\dagger : \left(\frac{\mathcal{C}}{}\right) \to \left(\frac{{}^\bullet I\mathcal{C}}{}\right)$. It is in fact defined more generally on a category whose objects are surjections of categories $\mathcal{E} \to \mathcal{C}$ (without necessarily requiring that the surjection be part of an extension) and where the morphisms are commutative squares. It is really part of the construction of the acyclic complex given in the last section. Thus given a surjection of categories $\mathcal{E} \to \mathcal{C}$ which is the identity on objects we let $N = \mathrm{Ker}(R\mathcal{E} \to R\mathcal{C})$ and construct the short exact sequence of $R\mathcal{C}$-modules

$$(B|\mathcal{E}^\dagger) = 0 \to \frac{N}{N \cdot {}^\bullet I\mathcal{E}} \to \frac{{}^\bullet I\mathcal{E}}{N \cdot {}^\bullet I\mathcal{E}} \to {}^\bullet I\mathcal{C} \to 0$$

where $B = N/(N \cdot {}^\bullet I\mathcal{E})$ and $\mathcal{E}^\dagger = {}^\bullet I\mathcal{E}/(N \cdot {}^\bullet I\mathcal{E})$. If the surjection is part of an extension $K \to \mathcal{E} \to \mathcal{C}$ we write $K^\dagger$ instead of B and we also write $(K|\mathcal{E})^\dagger = (K^\dagger|\mathcal{E}^\dagger)$. It is a straightforward check that $\dagger$ is functorial on morphisms of extensions.

In the opposite direction, given an extension of $R\mathcal{C}$-modules

$$(A|U) = 0 \to A \to U \xrightarrow{\phi} {}^\bullet I\mathcal{C} \to 0$$

we define an extension of categories

$$(A|U)^\dagger := (A^\dagger|U^\dagger) = A^\dagger \to U^\dagger \xrightarrow{\phi^\dagger} \mathcal{C}.$$

Here $U^\dagger$ is the category with the same objects as $\mathcal{C}$ and morphisms $x \to y$ are defined to be the pairs (u, α) for which $\alpha : x \to y$ in $\mathcal{C}$, $u \in 1_y \cdot U$ and $\phi(u) = \alpha - 1_{\mathrm{cod}(\alpha)}$. Composition of these morphisms is $(u, \alpha) \circ (v, \beta) = (u + {}^\alpha v, \alpha\beta)$. We may check that $\phi(u + {}^\alpha v) = \alpha - 1_{\mathrm{cod}(\alpha)} + \alpha(\beta - 1_{\mathrm{cod}(\beta)}) = \alpha\beta - 1_{\mathrm{cod}(\alpha\beta)}$ and that composition is associative. The surjection $\phi^\dagger : U^\dagger \to \mathcal{C}$ is defined to be the identity on objects and $\phi^\dagger(u, \alpha) = \alpha$ on morphisms. Also we define

$$A^\dagger(x) = \phi^{\dagger -1}(1_x) = \{(u, 1_x) \bigm| u \in U(x),\ \phi(u) = 0\}$$

and this is isomorphic to $A(x)$ via an isomorphism $(u, 1_x) \leftrightarrow u$. Observe that this isomorphism is functorial, so that $A^\dagger \cong A$ as $R\mathcal{C}$-modules. It is a straightforward check that $A^\dagger \to U^\dagger \to \mathcal{C}$ is indeed a category extension of $\mathcal{C}$ and that we have a second functor (denoted by the same symbol) $\dagger : \left(\frac{{}^\bullet I\mathcal{C}}{}\right) \to \left(\frac{\mathcal{C}}{}\right)$.

Note that the functor just described is in fact described by the pullback

$$\begin{array}{ccccc} A & \to & U^\dagger & \to & \mathcal{C} \\ & & \downarrow & & \downarrow \\ A & \to & U \rtimes \mathcal{C} & \to & {}^\bullet I\mathcal{C} \rtimes \mathcal{C} \end{array}$$

in the category of small categories, where $\mathcal{C} \to {}^\bullet I\mathcal{C} \rtimes \mathcal{C}$ is the functor $c \mapsto (c - 1_{\mathrm{cod}(c)}, c)$..

We set about showing that these two functors $\dagger$ between the extension categories are inverse equivalences, and start with some preliminary lemmas.

Lemma 3.2. *Let $\mathcal{E}$ and $\mathcal{C}$ be categories with the same objects and let $p : \mathcal{E} \to \mathcal{C}$ be a surjection of categories which is the identity on objects. Then* $\mathrm{Ker}(R\mathcal{E} \to R\mathcal{C})$ *is generated as an R-module by elements $w_1 - w_2$ where w_1 and w_2 are morphisms in $\mathcal{E}$ with $p(w_1) = p(w_2)$.*

Proof. The elements $w_1 - w_2$ certainly lie in the kernel. Let I be the R-submodule of $R\mathcal{E}$ generated by such $w_1 - w_2$. The quotient $R\mathcal{E}/I$ is spanned by the images $\bar{w} = w + I$ of morphisms w in $\mathcal{E}$ and if $p(w_1) = p(w_2)$ then $\bar{w}_1 = \bar{w}_2$, so if we pick $\alpha_0 \in p^{-1}(\alpha)$ for each $\alpha \in \mathcal{C}$ the $\bar{\alpha}_0$ span $R\mathcal{E}/I$. Their images in $R\mathcal{C}$ are independent, and $R\mathcal{C}$ is an image of $R\mathcal{E}/I$, so it follows that the $\bar{\alpha}_0$ are independent, and hence form a basis for $R\mathcal{E}/I$. The mapping $R\mathcal{E}/I \to R\mathcal{C}$ induces a bijection of bases and so is an isomorphism. Therefore I is equal to the kernel. □

We now give a smaller set of ideal generators of N.

Corollary 3.3. *Let $\mathcal{E}$ and $\mathcal{C}$ be categories with the same objects, let $p : \mathcal{E} \to \mathcal{C}$ be a surjection of categories which is the identity on objects and let $N =$* $\mathrm{Ker}(R\mathcal{E} \to R\mathcal{C})$. *Consider the set of pairs (w_1, w_2) of morphisms of $\mathcal{E}$ with the property that $p(w_1) = p(w_2)$ and write $(w_1, w_2) \le (w_1', w_2')$ if $w_1' = uw_1v$ and $w_2' = uw_2v$ for certain morphisms u and v in $\mathcal{E}$. Then the elements $w_1 - w_2$ where (w_1, w_2) is minimal with respect to $\le$ generate N as a 2-sided ideal.*

Note that the relation $\le$ is a preorder, and not necessarily a partial order.

Proof. If $w_1' = uw_1v$ and $w_2' = uw_2v$ then $w_1' - w_2' = u(w_1 - w_2)v$ lies in the ideal generated by $w_1 - w_2$. The result now follows from Lemma 3.2. □

We will use the following result in Proposition 3.5

Lemma 3.4. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories. Then $R\mathcal{E}$ is projective as an $R\mathcal{K}$-module.*

Proof. Since $\mathcal{K}$ is a disjoint union of groups $K(x)$ where x ranges over the objects of $\mathcal{K}$, it suffices to check for each x that $1_x \cdot R\mathcal{E} \cong \bigoplus_y R\,\mathrm{Hom}_{\mathcal{E}}(y,x)$ is projective as a $RK(x)$-module. However $K(x)$ permutes each set $\mathrm{Hom}_{\mathcal{E}}(y,x)$ freely (as is immediate from the definition of an extension), and so the module is indeed projective. □

Proposition 3.5. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories, let $N = \mathrm{Ker}(R\mathcal{E} \to R\mathcal{C})$ and suppose that $R = \mathbb{Z}$. Then*

(1) *N is generated as a right $R\mathcal{E}$-module by the elements $k_x - 1_x$ where x is an object of $\mathcal{K}$ and $1_x, k_x \in \mathrm{End}_{\mathcal{K}}(x)$. Thus $N = I\mathcal{K} \cdot R\mathcal{E}$.*
(2) *$H_1(\mathcal{K}) \cong N/(N \cdot {}^{\bullet}I\mathcal{E})$ as left $R\mathcal{C}$-modules.*
(3) *The extension $(\mathcal{K}|\mathcal{E})^{\dagger} = (\mathcal{K}^{\dagger}|\mathcal{E}^{\dagger})$ of $R\mathcal{C}$-modules takes the form*

$$0 \to H_1(\mathcal{K}) \to R\mathcal{C} \otimes_{R\mathcal{E}} {}^{\bullet}I\mathcal{E} \to {}^{\bullet}I\mathcal{C} \to 0,$$

the right hand map being induced by the surjection $\mathcal{E} \to \mathcal{C}$. Thus if the endomorphism groups in $\mathcal{K}$ are abelian we have $\mathcal{K}^{\dagger} \cong \mathcal{K}$ as $R\mathcal{C}$-modules.

Observe in statement (1) that because $\mathcal{K}$ is a disjoint union of groups the left and right augmentation ideals of $R\mathcal{K}$ coincide and we write simply $I\mathcal{K}$ for this ideal. In (2) the action of $R\mathcal{C}$ on $H_1(\mathcal{K})$ is the one specified in the remarks at the start of this section.

Proof. Observe that since $I\mathcal{K} \subseteq N$ we have $I\mathcal{K} \cdot R\mathcal{E} \subseteq N$. For the reverse inclusion, we know from Lemma 3.2 that N is generated as an R-module by elements $w_1 - w_2$ where $w_1, w_2 \in \mathcal{E}$ and $p(w_1) = p(w_2)$. Because we have an extension of categories there exists k in $\mathcal{K}$ so that $w_1 = kw_2$ and so $w_1 - w_2 = (k-1)w_2$ lies in the right submodule generated by the $k-1$. It follows that these elements as k ranges over $\mathcal{K}$ generate N as a right $R\mathcal{E}$-module, as was claimed in statement (1).

We now prove statement (2). Observe that for each $R\mathcal{K}$-module K we have $\underline{R} \otimes_{R\mathcal{K}} K \cong K/(I\mathcal{K} \cdot K)$. If K happens to be an $R\mathcal{E}$-module then

$$I\mathcal{K} \cdot K = I\mathcal{K} \cdot R\mathcal{E} \cdot K = N \cdot K$$

by part (1), so that $\underline{R}\otimes_{R\mathcal{K}} K \cong K/(N\cdot K)$. Since $R\mathcal{E}$ is projective as an $R\mathcal{K}$-module by Lemma 3.4 we may compute $\mathrm{Tor}^{R\mathcal{K}}(\underline{R}, \underline{R})$ by applying $\underline{R}\otimes_{R\mathcal{K}} -$ to either of the rows in the following diagram in which the vertical arrows are inclusions:

$$\begin{array}{ccccc} {}^{\bullet}I\mathcal{E} & \longrightarrow & R\mathcal{E} & \longrightarrow & \underline{R} \\ \uparrow & & \uparrow & & \| \\ I\mathcal{K} & \longrightarrow & R\mathcal{K} & \longrightarrow & \underline{R} \end{array}.$$

This gives a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Tor}^{R\mathcal{K}}(\underline{R}, \underline{R}) & \to & {}^{\bullet}I\mathcal{E}/(N\cdot{}^{\bullet}I\mathcal{E}) & \longrightarrow & R\mathcal{E}/N \\ & & \uparrow\cong & & \uparrow & & \uparrow \\ 0 & \to & \mathrm{Tor}^{R\mathcal{K}}(\underline{R}, \underline{R}) & \to & I\mathcal{K}/(I\mathcal{K})^2 & \xrightarrow{0} & \underline{R} \end{array}.$$

The kernel of the top right hand map may be identified as $N/(N\cdot{}^{\bullet}I\mathcal{E})$ so we obtain an isomorphism between this group and $\mathrm{Tor}^{R\mathcal{K}}(\underline{R}, \underline{R})$. It remains to show that this isomorphism is one of $R\mathcal{C}$-modules

Since $\mathcal{K}$ is a disjoint union of groups we may identify $\mathrm{Tor}^{R\mathcal{K}}(\underline{R}, \underline{R}) = H_1(\mathcal{K})$ as the direct product of the abelianizations $K(x)/K(x)'$ taken over the objects of $\mathcal{K}$, where we write $K(x) = \mathrm{Aut}_{\mathcal{K}}(x)$, and we have a well known isomorphism $H_1(\mathcal{K}) \to I\mathcal{K}/(I\mathcal{K})^2$ specified by

$$kK(x)' \mapsto k - 1 + (I\mathcal{K})^2.$$

Composing this with the map to $N/(N\cdot{}^{\bullet}I\mathcal{E})$ induced by the inclusion of $I\mathcal{K}$ we obtain an isomorphism $H_1(\mathcal{K}) \to N/(N\cdot{}^{\bullet}I\mathcal{E})$ specified by

$$kK(x)' \mapsto k - 1 + (N\cdot{}^{\bullet}I\mathcal{E})$$

where $k \in K(x)$. We show finally that this isomorphism is a homormorphism of $R\mathcal{E}$-modules, and hence of $R\mathcal{C}$-modules. If $w : x \to y$ is a morphism in $\mathcal{E}$ and $k \in K(x)$ then ${}^{w}k$ is defined by an equation $wk = {}^{w}kw$ and ${}^{w}kK(y)'$ maps to ${}^{w}k - 1_y + (N\cdot{}^{\bullet}I\mathcal{E})$. Now

$$\begin{aligned} {}^{w}k - 1_y &= ({}^{w}k - 1_y)w + ({}^{w}k - 1_y)(1 - w) \\ &= {}^{w}kw - w + ({}^{w}k - 1_y)(1 - w) \\ &= w(k - 1) + ({}^{w}k - 1_y)(1 - w) \end{aligned}$$

so that ${}^{w}k - 1_y + (N\cdot{}^{\bullet}I\mathcal{E}) = w(k - 1 + (N\cdot{}^{\bullet}I\mathcal{E}))$. This completes the proof of (2), and (3) is immediate from the construction of $(\mathcal{K}|\mathcal{E})^{\dagger}$. $\square$

From the definitions and by dimension shifting we know that

$$H^2(\mathcal{C}, K) = \mathrm{Ext}^2_{R\mathcal{C}}(\underline{R}, K) \cong \mathrm{Ext}^1_{R\mathcal{C}}({}^{\bullet}I\mathcal{C}, K)$$

so that equivalence classes of category extensions of $\mathcal{C}$ by K are in bijection with equivalence classes of module extensions of ${}^{\bullet}I\mathcal{C}$ by K. This bijection is compatible with the the equivalence of categories in the next theorem, in that if two extensions correspond under the equivalence then their cohomology classes correspond under the isomorphism of cohomology groups.

Theorem 3.6. *The extension categories* $\binom{\mathcal{C}}{}$ *and* $\binom{\bullet I\mathcal{C}}{}$ *are equivalent.*

Proof. We have already constructed functors in each direction between these two categories. We now show that given an extension of $R\mathcal{C}$-modules

$$(A|U) = 0 \to A \to U \xrightarrow{p} {}^\bullet I\mathcal{C} \to 0$$

and an extension of categories

$$(K|\mathcal{E}) = K \to \mathcal{E} \to \mathcal{C}$$

there are natural isomorphisms $(A|U) \cong (A|U)^{\dagger\dagger}$ and $(K|\mathcal{E}) \cong (K|\mathcal{E})^{\dagger\dagger}$.

Given an extension $(A|U)$ of ${}^\bullet I\mathcal{C}$ the category $U^\dagger$ has morphisms which are certain pairs (u, α) with $\alpha \in \mathcal{C}$ and now $U^{\dagger\dagger}$ is constructed as a quotient of ${}^\bullet I(U^\dagger)$, which is spanned by elements $(u, \alpha) - 1_{\mathrm{cod}(\alpha)}$. We get a morphism ${}^\bullet I(U^\dagger) \to U$ by $(u, \alpha) - 1_{\mathrm{cod}(\alpha)} \mapsto u$. This is an $RU^\dagger$-module homomorphism, and since the target is an $R\mathcal{C}$-module we obtain an $R\mathcal{C}$-module homomorphism $\eta : {}^\bullet I(U^\dagger)/(N_1 \cdot {}^\bullet I(U^\dagger)) \to U$, where $N_1 = \mathrm{Ker}(RU^\dagger \to R\mathcal{C})$, so that the following diagram commutes:

$$\begin{array}{cccccc}
(A|U)^{\dagger\dagger}: & \frac{N_1}{N_1 \cdot {}^\bullet I(U^\dagger)} & \to & \frac{{}^\bullet I(U^\dagger)}{N_1 \cdot {}^\bullet I(U^\dagger)} & \to & {}^\bullet I\mathcal{C} \\
 & \downarrow & & \downarrow \eta & & \| \\
(A|U): & A & \to & U & \to & {}^\bullet I\mathcal{C}
\end{array}.$$

The restriction of η to $N_1/(N_1 \cdot {}^\bullet I(U^\dagger))$ is the morphism determined by $(u, 1_y) - (0, 1_y) + N_1 \cdot {}^\bullet I(U^\dagger) \mapsto u$ where $u \in A(y)$ and we have seen in the proof of Proposition 3.5 that it is an isomorphism. We deduce that η is itself an isomorphism. We thus have an isomorphism $(A|U)^{\dagger\dagger} \to (A|U)$, and it is evident that our construction is natural.

Given an extension $(K|\mathcal{E})$ of $\mathcal{C}$ the category $\mathcal{E}^{\dagger\dagger}$ has morphisms of the form (v, β) where $v \in {}^\bullet I\mathcal{E}/(N \cdot {}^\bullet I\mathcal{E})$ and $\beta \in \mathcal{C}$. Given a morphism γ in $\mathcal{E}$ we define a functor $\mathcal{E} \to \mathcal{E}^{\dagger\dagger}$ which is the identity on objects, and which on morphisms is $\gamma \mapsto (1 - \gamma + N \cdot {}^\bullet I\mathcal{E}, p(\gamma))$. We may check that this is indeed a functor. We have seen in Proposition 3.5 that the restriction of this morphism to K is an isomorphism, and from this it follows that the functor $\mathcal{E} \to \mathcal{E}^\dagger$ is an isomorphism. We obtain in this way a natural isomorphism $(K|\mathcal{E}) \to (K|\mathcal{E})^{\dagger\dagger}$. □

We now consider the same constructions in a slightly different set-up where we do not suppose at the beginning that we have an extension of categories.

Theorem 3.7. *Let $p : \mathcal{G} \to \mathcal{C}$ be a surjection of categories where $\mathcal{G}$ and $\mathcal{C}$ have the same objects, and suppose the surjection is the identity on objects. Let $N = \mathrm{Ker}(R\mathcal{G} \to R\mathcal{C})$. Then there is a commutative diagram of functors between categories in which the bottom row is an extension*

$$\begin{array}{ccccc} & & \mathcal{G} & \to & \mathcal{C} \\ & & \downarrow & & \| \\ \frac{N}{N\cdot {}^\bullet I\mathcal{G}} & \to & \mathcal{G}^{\dagger\dagger} & \to & \mathcal{C} \end{array}$$

with the universal property that any diagram

$$\begin{array}{ccccc} & & \mathcal{G} & \to & \mathcal{C} \\ & & \downarrow & & \| \\ K & \to & \mathcal{E} & \to & \mathcal{C} \end{array},$$

in which the bottom row is an extension by an $R\mathcal{C}$-module K, factors uniquely as

$$\begin{array}{ccccc} & & \mathcal{G} & \to & \mathcal{C} \\ & & \downarrow & & \| \\ \frac{N}{N\cdot {}^\bullet I\mathcal{G}} & \to & \mathcal{G}^{\dagger\dagger} & \to & \mathcal{C} \\ \downarrow & & \downarrow & & \| \\ K & \to & \mathcal{E} & \to & \mathcal{C} \end{array}.$$

Another way to express this is to regard the functors † as being defined on categories where the objects are surjections of categories $\mathcal{G} \to \mathcal{C}$ which are the identity on objects, on the one hand, and surjections of $R\mathcal{C}$-modules $U \to {}^\bullet I\mathcal{C}$ on the other hand and in which the morphisms are commutative squares. The functor which takes categories to modules is right adjoint to the functor which takes modules to categories.

Proof. Given the surjection $p : \mathcal{G} \to \mathcal{C}$ we start with the functorial construction described between Proposition 3.1 and Lemma 3.2 of an extension of $R\mathcal{C}$-modules

$$0 \to B \to \mathcal{G}^\dagger \to {}^\bullet I\mathcal{C} \to 0$$

where $B = N/(N \cdot {}^\bullet I\mathcal{G})$ and $\mathcal{G}^\dagger = {}^\bullet I\mathcal{G}/(N \cdot {}^\bullet I\mathcal{G})$. We now apply the second functor $\dagger : \left(\frac{{}^\bullet I\mathcal{C}}{}\right) \to \left(\frac{\mathcal{C}}{}\right)$ to get an extension which is the top row of the following diagram in which the square is a pullback

$$\begin{array}{ccccc} B & \to & \mathcal{G}^{\dagger\dagger} & \to & \mathcal{C} \\ & & \downarrow & & \downarrow \\ & & \mathcal{G}^{\dagger} \rtimes \mathcal{C} & \to & {}^{\bullet}I\mathcal{C} \rtimes \mathcal{C} \end{array}.$$

The functor $\mathcal{G} \to \mathcal{G}^{\dagger} \rtimes \mathcal{C} = \frac{{}^{\bullet}I\mathcal{G}}{N \cdot {}^{\bullet}I\mathcal{G}} \rtimes \mathcal{C}$ specified by

$$\alpha \mapsto (\alpha - 1_{\mathrm{cod}(\alpha)} + N \cdot {}^{\bullet}I\mathcal{G}, p(\alpha))$$

does lift to a functor $\mathcal{G} \to \mathcal{G}^{\dagger\dagger}$ so that everything commutes, by the pullback property. This establishes the existence of the first diagram in the statement of the theorem.

A commutative diagram of functors

$$\begin{array}{cccccc} & & & \mathcal{G} & \to & \mathcal{C} \\ & & & \downarrow & & \| \\ (K|\mathcal{E}): & K & \to & \mathcal{E} & \to & \mathcal{C} \end{array}$$

in which the bottom row is an extension does give rise to a commutative diagram of $R\mathcal{C}$-modules

$$\begin{array}{cccccccccc} (B|\mathcal{G}^{\dagger}): & 0 & \to & B & \to & \mathcal{G}^{\dagger} & \to & {}^{\bullet}I\mathcal{C} & \to & 0 \\ & & & \downarrow & & \downarrow & & \| & & \\ (K|\mathcal{E})^{\dagger}: & 0 & \to & K & \to & \mathcal{E}^{\dagger} & \to & {}^{\bullet}I\mathcal{C} & \to & 0 \end{array}$$

by functoriality of the † construction, and hence a commutative diagram

$$\begin{array}{cccccc} (B|\mathcal{G}^{\dagger})^{\dagger} = (B|\mathcal{G}^{\dagger\dagger}): & B & \to & \mathcal{G}^{\dagger\dagger} & \to & \mathcal{C} \\ & \downarrow & & \downarrow & & \| \\ (K|\mathcal{E}) = (K|\mathcal{E})^{\dagger\dagger}: & K & \to & \mathcal{E} & \to & \mathcal{C} \end{array}$$

by the equivalence of extension categories in Theorem 3.6. From the construction we may verify that the composite $\mathcal{G} \to \mathcal{G}^{\dagger\dagger} \to \mathcal{E}$ coincides with the original morphism $\mathcal{G} \to \mathcal{E}$.

Any morphism of extensions $(B|\mathcal{G}^{\dagger\dagger}) \to (K|\mathcal{E})$ which appears in a factorization of $\mathcal{G} \to \mathcal{E}$ corresponds under the equivalence of categories to a morphism of extensions $(B|\mathcal{G}^{\dagger}) \to (K|\mathcal{E}^{\dagger})$ in which the morphism $\mathcal{G}^{\dagger} \to \mathcal{E}^{\dagger}$ is the one induced from $\mathcal{G} \to \mathcal{E}$. Since this morphism is uniquely determined, the original morphism of extensions is uniquely determined. □

4. Relation Modules for Categories

Relation modules for groups have been extensively studied by many authors, and we refer to [10] for an indication of the depth and intricacy of the theory. In this section we make a new definition of relation modules for small categories in general and establish some of their immediate properties, bearing in mind that it is not even clear how the definition should be made outside the realm of groups. The properties we establish are that relation modules occur as the kernels in certain projective objects in the extension categories $\left(\frac{\mathcal{C}}{}\right)$ and $\left(\frac{{}^{\bullet}I\mathcal{C}}{}\right)$, and also that the definition we make here genuinely extends the definition usually made for groups. It would be possible to develop the theory of relation modules of categories further along the lines of [10], but we do not attempt this here. Since relation modules are naturally defined over the integers, in this section we work over the ring $\mathbb{Z}$ instead of a more general ring R.

First we recall that given a presentation of groups

$$1 \to K \to F \to G \to 1$$

(that is, a short exact sequence of groups in which F is a free group) the relation module is defined to be the abelianization K/K', regarded as a $\mathbb{Z}G$-module using conjugation within F.

To define a relation module for a small category $\mathcal{C}$ we start with a surjective functor $\mathcal{F} \to \mathcal{C}$ where $\mathcal{F}$ is a free category with the same objects as $\mathcal{C}$, and such that the functor is the identity on objects. At this point we do not immediately have the means to copy the construction from group theory, since we do not have a satisfactory notion of the kernel of the functor. Instead we define the associated *relation module* to be the $\mathbb{Z}\mathcal{C}$-module $B = N/(N \cdot {}^{\bullet}I\mathcal{F})$ where $N = \operatorname{Ker}(\mathbb{Z}\mathcal{F} \to \mathbb{Z}\mathcal{C})$. This is a special case of the module considered in Section 3 and it appears in the short exact sequence of $\mathbb{Z}\mathcal{C}$-modules

$$(B|\mathcal{F}^{\dagger}) = 0 \to \frac{N}{N \cdot {}^{\bullet}I\mathcal{F}} \to \mathcal{F}^{\dagger} \to {}^{\bullet}I\mathcal{C} \to 0$$

obtained by applying the functor $\dagger$ to $\mathcal{F} \to \mathcal{C}$ as described after Proposition 3.1. This sequence is also part of the resolution constructed in Theorem 2.3.

Proposition 4.1. *Let $\mathcal{F} \to \mathcal{C}$ be a surjection of categories in which $\mathcal{F}$ is a free category with the same objects as $\mathcal{C}$, and let $N = \operatorname{Ker}(\mathbb{Z}\mathcal{F} \to \mathbb{Z}\mathcal{C})$. Then the extension of $\mathbb{Z}\mathcal{C}$-modules*

$$(B|\mathcal{F}^{\dagger}) = 0 \to \frac{N}{N \cdot {}^{\bullet}I\mathcal{F}} \to \mathcal{F}^{\dagger} \to {}^{\bullet}I\mathcal{C} \to 0$$

is a projective object in the extension category $\left(\frac{\bullet I\mathcal{C}}{}\right)$*, and the corresponding extension*

$$(B|\mathcal{F}^{\dagger\dagger}) = \frac{N}{N\cdot \bullet I\mathcal{F}} \to \mathcal{F}^{\dagger\dagger} \to \mathcal{C}$$

is a projective object in $\left(\frac{\mathcal{C}}{}\right)$*.*

Proof. The first statement follows from the fact that $\mathcal{F}^\dagger$ is a projective $\mathbb{Z}\mathcal{C}$-module as in the proof of Theorem 2.3. For, given any diagram of extensions

$$\begin{array}{ccc} & & (B|\mathcal{F}^\dagger) \\ & & \downarrow \\ (A_1|E_1) & \to & (A_2|E_2) \end{array}$$

in which the horizontal map is an epimorphism we obtain a diagram of $\mathbb{Z}\mathcal{C}$-modules

$$\begin{array}{ccc} & & \mathcal{F}^\dagger \\ & & \downarrow \\ E_1 & \to & E_2 \end{array}$$

in which the horizontal map is a surjection, and now the lift morphism $\mathcal{F}^\dagger \to E_1$ gives rise to a map of extensions $(B|\mathcal{F}^\dagger) \to (A_1|E_1)$ which is the required lift.

To show that the extension of categories $(B|\mathcal{F}^\dagger)$ is projective, the easiest thing at this point is to say that it follows from the equivalence of categories in Theorem 3.6. It is also possible to give a direct argument to show that we can lift through epimorphisms of extensions, using the fact that $\mathcal{F}$ is a free category together with the universal property of Theorem 3.7, but this argument is longer. □

There is a difference between the definition of a relation module for a group and a relation module for a category, in that the first uses a surjection from a free group and the second uses a surjection from a free category. In case we are presenting a group, the free category is in fact a free monoid since there is only one object. We show now that the category definition gives a module which is a relation module in the sense using free groups. We also show that if we consider a set of group generators for G which do not generate G as a monoid (something which only happens when there are generators of infinite order) then the relation module in the sense of group theory is obtained from the category theory relation module by inducing it from the submonoid of G generated by the elements, to G. Thus relation modules for a group (defined using free groups) are in fact computable using free monoids. This is the content of the next result.

Proposition 4.2. *Let F be a free group on d generators and let $\mathcal{F}$ be the (free) submonoid of F generated by these generators. Let $\phi : F \to G$ be a surjective group homomorphism to a group G and let $\mathcal{C} = \phi(\mathcal{F})$ be the submonoid of G which is the image of $\mathcal{F}$ under ϕ. Let $N = \mathrm{Ker}(\mathbb{Z}\mathcal{F} \to \mathbb{Z}G)$, and let $K = \mathrm{Ker}(F \to G)$. The sequence*

$$(B|\mathbb{Z}\mathcal{C}^d) = 0 \to \frac{N}{N \cdot {}^{\bullet}I\mathcal{F}} \to \mathbb{Z}\mathcal{C}^d \to {}^{\bullet}I\mathcal{C} \to 0$$

has the property that $\mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} (B|\mathbb{Z}\mathcal{C}^d)$ is isomorphic to the sequence of $\mathbb{Z}G$-modules

$$0 \to K/K' \to \mathbb{Z}G^d \to IG \to 0$$

coming from the group presentation. In particular, $\mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} {}^{\bullet}I\mathcal{C} \cong IG$ and $\mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} B \cong K/K'$ as $\mathbb{Z}G$-modules.

Proof. We know from Proposition 2.2 that ${}^{\bullet}I\mathcal{F} \cong \mathbb{Z}\mathcal{F}^d$ is free of rank d on generators $1 - s$ where s ranges through the generators of $\mathcal{F}$, and in the sequence

$$0 \to N \to {}^{\bullet}I\mathcal{F} \to {}^{\bullet}I\mathcal{C} \to 0$$

the map ${}^{\bullet}I\mathcal{F} \to {}^{\bullet}I\mathcal{C}$ sends $1 - s$ to $1 - \phi(s)$. The sequence $(B|\mathbb{Z}\mathcal{C}^d)$ is obtained from this by factoring out $N \cdot {}^{\bullet}I\mathcal{F}$. The functor $\mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} -$ is exact (see [5, Ch. X, proof of 4.1]), and we now apply it to this sequence and also the sequence

$$0 \to {}^{\bullet}I\mathcal{C} \to \mathbb{Z}\mathcal{C} \to \mathbb{Z} \to 0.$$

Since $\mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} \mathbb{Z} \cong \mathbb{Z}$ (we leave this to the reader) we get an exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} {}^{\bullet}I\mathcal{C} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

from which we deduce that

$$\mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} {}^{\bullet}I\mathcal{C} \cong IG$$

via an isomorphism which associates $1 \otimes (1 - s)$ to $1 - s$. In the exact sequence

$$0 \to \mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} \frac{N}{N \cdot {}^{\bullet}I\mathcal{F}} \to \mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} \mathbb{Z}\mathcal{C}^d \to \mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} {}^{\bullet}I\mathcal{C} \to 0$$

we now deduce that the right hand mapping identifies with the right hand mapping of the sequence coming from the group presentation. It follows that these two sequences are isomorphic, and hence also that the modules are isomorphic as claimed. □

Corollary 4.3. *Let F be a free group on d generators and let $\mathcal{F}$ be the (free) submonoid of F generated by these generators. Let $\phi : F \to G$ be a group homomorphism to a group G such that the images of the generators of F generate G as a monoid (so $G = \phi(\mathcal{F})$). Let $N = \mathrm{Ker}(\mathbb{Z}\mathcal{F} \to \mathbb{Z}G)$, and let $K = \mathrm{Ker}(F \to G)$. Then, as $\mathbb{Z}G$-modules, the relation module computed from $F \to G$ is isomorphic to the relation module computed from $\mathcal{F} \to G$. Furthermore, the category extension*

$$\frac{N}{N \cdot \bullet I\mathcal{F}} \to \mathcal{F}^{\dagger\dagger} \to G$$

coming from the monoid presentation is isomorphic to the extension $K/K' \to F/K' \to G$ coming from the group presentation $F \to G$.

Proof. We apply Proposition 4.2 in the case $\mathcal{C} = G$, so that the functor $\mathbb{Z}G \otimes_{\mathbb{Z}\mathcal{C}} -$ is the identity. This gives the isomorphism of relation modules and also shows that the two $\mathbb{Z}\mathcal{C}$-module extensions of IG coming from the monoid presentation and from the group presentation are isomorphic.

To show that the two extensions of categories are isomorphic we may use the equivalence of extension categories in Theorem 3.6. This completes the proof, but as an alternative we may observe that the diagram

$$\begin{array}{ccccc} & & \mathcal{F} & \longrightarrow & G \\ & & \big\downarrow & & \| \\ K/K' & \longrightarrow & F/K' & \longrightarrow & G \end{array}$$

satisfies the universal property of Theorem 3.7. For, suppose we have a diagram

$$\begin{array}{ccccc} & & \mathcal{F} & \to & G \\ & & \big\downarrow & & \| \\ M & \to & \mathcal{E} & \to & G \end{array}$$

in which the bottom row is an extension of categories by a $\mathbb{Z}G$-module M. Necessarily $\mathcal{E}$ must be a group (we leave this easy argument to the reader) and so the morphism $\mathcal{F} \to \mathcal{E}$ factors through the free group F as $\mathcal{F} \to F \to \mathcal{E}$ and K' is contained in the kernel of $F \to \mathcal{E}$ since M is abelian. Thus we obtain a factorization

$$\begin{array}{ccccc} & & \mathcal{F} & \to & G \\ & & \downarrow & & \parallel \\ K/K' & \to & F/K' & \to & G\,. \\ \downarrow & & \downarrow & & \parallel \\ M & \to & \mathcal{E} & \to & G \end{array}$$

which is unique since the images of the generators of $\mathcal{F}$ generate F/K' as a group. Since the other category extension in the statement of the corollary is also known to have this universal property by Theorem 3.7, the two category extensions are isomorphic. □

5. Five-Term Exact Sequences

Exact sequences of low-dimensional cohomology groups are presented in every thorough treatment of group cohomology. We present two pairs of five-term sequences in this section, one pair associated to representations of an extension of categories, the other pair associated to representations of the opposite of an extension of categories. The first of these is what we will use in this paper and we start by describing it, leaving the second possibility to the end of this section.

Theorem 5.1. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories, let B be a right $\mathbb{Z}\mathcal{C}$-module and let A a left $\mathbb{Z}\mathcal{C}$-module. There are exact sequences*

$$\begin{aligned} H_2(\mathcal{E}, B) \to H_2(\mathcal{C}, B) \to & \\ & B \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K}) \to H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, B) \to 0 \end{aligned}$$

and

$$\begin{aligned} H^2(\mathcal{E}, A) \leftarrow H^2(\mathcal{C}, A) \leftarrow & \\ & \mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(H_1(\mathcal{K}), A) \leftarrow H^1(\mathcal{E}, A) \leftarrow H^1(\mathcal{C}, A) \leftarrow 0. \end{aligned}$$

In this result $H_1(\mathcal{K})$ is the product of the abelianizations of the groups $K(x) = \mathrm{End}_{\mathcal{K}}(x)$ taken over the objects x of $\mathcal{K}$, and because we have an extension of categories it becomes a left $\mathbb{Z}\mathcal{C}$-module, as described at the start of Section 3.

Proof. We follow page 202 of [12]. We start with the short exact sequence of $\mathbb{Z}\mathcal{C}$-modules of Proposition 3.5

$$0 \to H_1(\mathcal{K}) \to \mathbb{Z}\mathcal{C} \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E} \to {}^\bullet I\mathcal{C} \to 0$$

and apply $B \otimes_{\mathbb{Z}\mathcal{C}} -$ to get an exact sequence

$$\begin{aligned} &\mathrm{Tor}^{\mathbb{Z}\mathcal{C}}(B, \mathbb{Z}\mathcal{C} \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E}) \to \mathrm{Tor}^{\mathbb{Z}\mathcal{C}}(B, {}^\bullet I\mathcal{C}) \\ &\qquad \to B \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K}) \to B \otimes_{\mathbb{Z}\mathcal{C}} \mathbb{Z}\mathcal{C} \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E} \to B \otimes_{\mathbb{Z}\mathcal{C}} {}^\bullet I\mathcal{C} \to 0. \end{aligned}$$

In this we can identify the terms

$$\mathrm{Tor}^{\mathbb{Z}\mathcal{C}}(B, {}^\bullet I\mathcal{C}) \cong H_2(\mathcal{C}, B)$$

and

$$B \otimes_{\mathbb{Z}\mathcal{C}} \mathbb{Z}\mathcal{C} \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E} \cong B \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E}.$$

We construct an epimorphism

$$H_2(\mathcal{E}, B) \to \mathrm{Tor}^{\mathbb{Z}\mathcal{C}}(B, \mathbb{Z}\mathcal{C} \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E})$$

as follows: take a short exact sequence

$$0 \to B_1 \to P \to B \to 0$$

of right modules, where P is a projective $\mathbb{Z}\mathcal{C}$-module, and apply the functors $- \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E}$ and $- \otimes_{\mathbb{Z}\mathcal{C}} (\mathbb{Z}\mathcal{C} \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E})$, which on $\mathbb{Z}\mathcal{C}$-modules give isomorphic results. We get a commutative diagram

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ H_2(\mathcal{E}, B) & \cong & \mathrm{Tor}^{\mathbb{Z}\mathcal{E}}(B, {}^\bullet I\mathcal{E}) & \to & \mathrm{Tor}^{\mathbb{Z}\mathcal{C}}(B, \mathbb{Z}\mathcal{C} \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E}) \\ & & \downarrow & & \downarrow \\ & & B_1 \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E} & = & B_1 \otimes_{\mathbb{Z}\mathcal{C}} (\mathbb{Z}\mathcal{C} \otimes {}^\bullet I\mathcal{E}) \\ & & \downarrow & & \downarrow \\ & & P \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E} & = & P \otimes_{\mathbb{Z}\mathcal{C}} (\mathbb{Z}\mathcal{C} \otimes {}^\bullet I\mathcal{E}) \end{array}$$

where the vertical morphisms are part of long exact Tor sequences. The top term is zero because P is projective and we see from this that the top arrow from left to right exists and is an epimorphism.

At this point we have an exact sequence

$$H_2(\mathcal{E}, B) \to H_2(\mathcal{C}, B) \to B \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K}) \to B \otimes_{\mathbb{Z}\mathcal{E}} {}^\bullet I\mathcal{E} \to B \otimes_{\mathbb{Z}\mathcal{C}} {}^\bullet I\mathcal{C} \to 0.$$

We can replace the two right non-zero terms by $H_1(\mathcal{E}, B)$ and $H_1(\mathcal{C}, B)$ in the following way. We apply $B \otimes_{\mathbb{Z}\mathcal{E}} -$ to the diagram

$$\begin{array}{ccc} {}^{\bullet}I\mathcal{E} & \longrightarrow & {}^{\bullet}I\mathcal{C} \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}\mathcal{E} & \longrightarrow & \mathbb{Z}\mathcal{C} \\ \big\downarrow & & \big\downarrow \\ \underline{\mathbb{Z}} & = & \underline{\mathbb{Z}} \end{array}$$

to get a commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & \\ \big\downarrow & & \big\downarrow & & \\ H_1(\mathcal{E}, B) & \longrightarrow & H_1(\mathcal{C}, B) & & \\ \big\downarrow & & \big\downarrow & & \\ B \otimes_{\mathbb{Z}\mathcal{E}} {}^{\bullet}I\mathcal{E} & \longrightarrow & B \otimes_{\mathbb{Z}\mathcal{C}} {}^{\bullet}I\mathcal{C} & = & B \otimes_{\mathbb{Z}\mathcal{E}} {}^{\bullet}I\mathcal{C} \\ \big\downarrow & & \big\downarrow & & \\ B \otimes_{\mathbb{Z}\mathcal{E}} \mathbb{Z}\mathcal{E} & \cong & B \otimes_{\mathbb{Z}\mathcal{E}} \mathbb{Z}\mathcal{C} & \cong & B \\ \big\downarrow & & \big\downarrow & & \\ B \otimes_{\mathbb{Z}\mathcal{E}} \underline{\mathbb{Z}} & = & B \otimes_{\mathbb{Z}\mathcal{E}} \underline{\mathbb{Z}} & & \end{array}$$

with exact columns from which we deduce that the morphism $H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, B)$ is surjective and

$$\operatorname{Ker}(B \otimes_{\mathbb{Z}\mathcal{E}} {}^{\bullet}I\mathcal{E} \to B \otimes_{\mathbb{Z}\mathcal{C}} {}^{\bullet}I\mathcal{C}) = \operatorname{Ker}(H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, B)).$$

Thus we obtain the sequence in the statement of the theorem.

The construction of the cohomology sequence is entirely analogous using the functor $\operatorname{Hom}_{\mathbb{Z}\mathcal{C}}(-, A)$ and we omit the details. □

We now report on two five-term exact sequences which arise when we have the opposite of an extension of categories. There is an asymmetry in the definition of an extension of categories $\mathcal{K} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}$ given at the start of Section 3, in that condition (2) of that definition contains an arbitrary choice which when reversed gives condition (2^{op}) in what follows.

(1) $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{C}$ all have the same objects, i and p are the identity on objects, i is injective on morphisms, and p is surjective on morphisms;

(2^{op}) whenever f and g are morphisms in $\mathcal{E}$ then $p(f) = p(g)$ if and only if there exists a morphism $k \in \mathcal{K}$ for which $f = gi(k)$. In that case, the morphism k is required to be unique.

We will call a pair of functors satisfying (1) and (2^{op}) an *opposite extension*, and it is equivalent to the requirement that the corresponding functors between the opposite categories form an extension.

We have already seen in Section 3 that given an extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$, $\mathcal{K}$ can be regarded as a functor $\mathcal{E} \to$ Groups. It is immediate that given an opposite extension we may regard $\mathcal{K}^{\mathrm{op}}$ as a functor $\mathcal{E}^{\mathrm{op}} \to$ Groups, and in fact since $\mathcal{K}$ is a disjoint union of groups we have $\mathcal{K} \cong \mathcal{K}^{\mathrm{op}}$ via the isomorphism which sends each morphism to its inverse. We see from all this that when we have an opposite extension $H_1(\mathcal{K})$ acquires the structure of a left $\mathbb{Z}\mathcal{C}^{\mathrm{op}}$-module, which is the same thing as a *right* $\mathbb{Z}\mathcal{C}$-module. More generally, it is the case that given an opposite extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$, a left $\mathbb{Z}\mathcal{E}$-module A and a right $\mathbb{Z}\mathcal{E}$-module B, the cohomology groups $H^q(\mathcal{K}, A)$ acquire the structure of left $\mathbb{Z}\mathcal{C}$-modules and the homology groups $H_q(\mathcal{K}, B)$ acquire the structure of right $\mathbb{Z}\mathcal{C}$-modules.

We now quote a result of Fei Xu [20] which generalizes the Lyndon-Hochschild-Serre spectral sequences in the cohomology of groups. Our notation is slightly different to Xu's, in that he defines homology groups $H_q(\mathcal{E}, B)$ when B is a left $\mathbb{Z}\mathcal{E}$-module, whereas in this paper we define these homology groups when B is a right $\mathbb{Z}\mathcal{E}$-module.

Theorem 5.2. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an opposite extension of categories, let B be a right $\mathbb{Z}\mathcal{E}$-module and A a left $\mathbb{Z}\mathcal{E}$-module. There are spectral sequences whose second pages are*

$$E^2_{p,q} = H_p(\mathcal{C}, H_q(\mathcal{K}, B)) \Rightarrow H_{p+q}(\mathcal{E}, B)$$

and

$$E_2^{p,q} = H^p(\mathcal{C}, H^q(\mathcal{K}, A)) \Rightarrow H^{p+q}(\mathcal{E}, A).$$

Just as in group cohomology we may deduce the five-term exact sequences from the Lyndon-Hochschild-Serre spectral sequences so here we may also deduce five-term exact sequences.

Corollary 5.3. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an opposite extension of categories, let B be a right $\mathbb{Z}\mathcal{E}$-module and A a left $\mathbb{Z}\mathcal{E}$-module. There are exact sequences*

$$\begin{aligned} H_2(\mathcal{E}, B) \to H_2(\mathcal{C}, H_0(\mathcal{K}, B)) \to \\ H_0(\mathcal{C}, H_1(\mathcal{K}, B)) \to H_1(\mathcal{E}, B) \to H_1(\mathcal{C}, H_0(\mathcal{K}, B)) \to 0 \end{aligned}$$

and

$$H^2(\mathcal{E},A) \leftarrow H^2(\mathcal{C},H^0(\mathcal{K},A)) \leftarrow \\ H^0(\mathcal{C},H^1(\mathcal{K},A)) \leftarrow H^1(\mathcal{E},A) \leftarrow H^1(\mathcal{C},H^0(\mathcal{K},A)) \leftarrow 0.$$

For comparison with Theorem 5.1 observe that

$$H_0(\mathcal{K},B) = B \otimes_{\mathbb{Z}\mathcal{K}} \underline{\mathbb{Z}} \cong \varinjlim B \cong \bigoplus_{x\in \mathrm{Ob}\mathcal{K}} H_0(\mathrm{End}_{\mathcal{K}}(x), B(x))$$

and

$$H^0(\mathcal{K},A) = \mathrm{Hom}_{\mathbb{Z}\mathcal{K}}(\underline{\mathbb{Z}},A) \cong \varprojlim A \cong \bigoplus_{x\in \mathrm{Ob}\mathcal{K}} H^0(\mathrm{End}_{\mathcal{K}}(x), A(x)).$$

since $\mathcal{K}$ is a disjoint union of groups. These groups have the structure of $\mathbb{Z}\mathcal{C}$-modules, and in the direct sum decompositions given on the right, the term corresponding to x is the image of 1_x on the module. In the special case that A and B are themselves representations of $\mathcal{C}$, made into representations of $\mathcal{E}$ via the given surjection to $\mathcal{C}$, these expressions simplify to $H_0(\mathcal{K},B) \cong B$ and $H^0(\mathcal{K},A) \cong A$. In this case also we have isomorphisms $H_1(\mathcal{K},B) \cong H_1(\mathcal{K}) \otimes_{\mathbb{Z}\mathcal{K}} B$ and $H^1(\mathcal{K},A) \cong \mathrm{Hom}_{\mathbb{Z}\mathcal{K}}(H_1(\mathcal{K}),A)$, by the Universal Coefficient Theorem 2.5.

6. The Schur Multiplier of a Category

We define the *Schur multiplier* of $\mathcal{C}$ to be $H_2(\mathcal{C}) = H_2(\mathcal{C},\underline{\mathbb{Z}})$, suppressing the constant coefficients $\underline{\mathbb{Z}}$ from the notation in homology and cohomology groups. It has properties which generalize the corresponding familiar results for groups and we now present the first of these.

Theorem 6.1. *Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension of categories and suppose that the induced homomorphism $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism. Then $\varinjlim H_1(\mathcal{K}) = \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K})$ is a homomorphic image of $H_2(\mathcal{C})$.*

Proof. This is an immediate consequence of the five-term exact homology sequence (Theorem 5.1). We take $B = \underline{\mathbb{Z}}$ so that the last four terms of the sequence are

$$H_2(\mathcal{C}) \to \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K}) \to H_1(\mathcal{E}) \to H_1(\mathcal{C}) \to 0$$

and the result is immediate on observing that $\underline{\mathbb{Z}} \otimes_{\mathbb{Z}\mathcal{C}} H_1(\mathcal{K}) \cong \varinjlim H_1(\mathcal{K})$. □

We take a moment to explain the connection between the above result and a familiar result in group theory. Suppose we have a group E with a normal subgroup K and factor group $C = E/K$. The condition $K \subseteq E'$ is equivalent to our condition on first homology that the induced homomorphism $E/E' \to C/C'$ be an isomorphism. Furthermore, the group $\varinjlim H_1(\mathcal{K})$ is isomorphic to $K/[K,E]$, the largest quotient of K on which E acts trivially. To require that the extension be central ($K \subseteq Z(E)$) is the same as to require $[K,E] = 1$. The result on the Schur multiplier of a group which Theorem 6.1 generalizes is sometimes stated in the form that if $K \subseteq E' \cap Z(E)$ then K is an image of the multiplier. This is seen to be a particular case of Theorem 6.1.

Group extensions for which $K \subseteq E'$ are sometimes called *stem extensions*, and as a generalization of this terminology we use the same term for extensions of categories in which $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism. We will study them further in Section 8. We are also going to discuss in greater depth the generalization to categories of the notion of a central extension, but for now we can take it to be an extension by a constant functor. If A is any abelian group we use the notation $\underline{A}$ for the constant functor on $\mathcal{C}$ with group A. It is well known that if G is a finite perfect group it has a universal central stem extension which is unique up to isomorphism. We now give a version of this result for categories as follows. It makes use of the Universal Coefficient Theorem 2.5, and not the five term exact sequence. The result will be strengthened in Theorem 8.6 where we show that the unique extension about to be described is in fact a stem extension.

Theorem 6.2. *Let $\mathcal{C}$ be a category for which $H_1(\mathcal{C})$ is a free abelian group and let A be an abelian group. There is an extension of categories $\underline{H_2(\mathcal{C})} \to \mathcal{E} \to \mathcal{C}$ with the property that any extension of $\mathcal{C}$ by the constant functor $\underline{A}$ is obtained from it as an explicit pushout, and such an extension with $\underline{H_2(\mathcal{C})}$ as its left-hand term is unique up to isomorphism if $H_2(\mathcal{C})$ is finitely generated.*

Proof. Since $\operatorname{Ext}(H_1(\mathcal{C}), A) = 0$ we have a functorial isomorphism $\theta : H^2(\mathcal{C}, \underline{A}) \to \operatorname{Hom}(H_2(\mathcal{C}), A)$ by the Universal Coefficients Theorem 2.5. Let $(\underline{H_2(\mathcal{C})}|\mathcal{E})$ be the extension of $\mathcal{C}$ by $\underline{H_2(\mathcal{C})}$ which corresponds to $1_{H_2(\mathcal{C})}$. We show this has the desired property. For, any extension by $\underline{A}$ corresponds to a homomorphism $f : H_2(\mathcal{C}) \to A$ and is the explicit pushout of $\underline{f} : \underline{H_2(\mathcal{C})} \to \underline{A}$ since the following diagram commutes:

$$\begin{array}{ccccc} H^2(\mathcal{C}, \underline{H_2(\mathcal{C})}) & \xrightarrow{\theta} & \operatorname{Hom}(H_2(\mathcal{C}), H_2(\mathcal{C})) & & 1 \\ \downarrow{\scriptstyle \underline{f}_*} & & \downarrow{\scriptstyle f_*} & & \downarrow \\ H^2(\mathcal{C}, \underline{A}) & \xrightarrow{\theta} & \operatorname{Hom}(H_2(\mathcal{C}), A) & & f \end{array}.$$

If we have another extension by $\underline{H_2(\mathcal{C})}$ with the same property we have maps

$$\begin{array}{ccccc} H_2(\mathcal{C}) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{C} \\ {\scriptstyle\alpha}\big\downarrow\big\uparrow{\scriptstyle\beta} & & \big\downarrow\big\uparrow & & \| \\ H_2(\mathcal{C}) & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{C} \end{array}$$

and now $1 = \beta_*\alpha_*1 = \beta\alpha$. From this it follows that β is surjective, and since $H_2(\mathcal{C})$ is finitely generated β must be an isomorphism. Therefore the extensions are isomorphic, by Proposition 3.1. □

7. Sublocally Constant Representations

We continue with our goal of showing that the role of the Schur multiplier in group theory may be duplicated in the context of categories. Prior to doing this we examine in this section some different ways in which the notion of a central group extension may be generalized.

The most obvious kind of category extension to consider which generalizes central group extensions is that of an extension by a constant functor, since on such functors the action of the category is trivial. In fact this notion is probably too restrictive to be useful. For example, the category we represent may be disconnected and there is no reason why we should only consider representations which are are forced to be the same on different components. To remedy this we say that a representation M of a category $\mathcal{C}$ is *locally constant* if and only if whenever $\alpha : x \to y$ is a morphism in $\mathcal{C}$ then $M(x) = M(y)$ and $M(\alpha)$ is the identity morphism. Thus if M is locally constant the groups $M(x)$ are constant on the connected components of $\mathcal{C}$.

There are still more general classes of representations which extend the notion of a trivial group representation and we define one of these now. We say that a representation M of $\mathcal{C}$ is *sublocally constant* if M is a subfunctor of a locally constant functor. These representations have a technical advantage, which is that each representation M has a largest sublocally constant quotient, whose direct limit equals $\varinjlim M$. We collect some of the first properties of sublocally constant functors

Lemma 7.1. *Let M be a representation of $\mathcal{C}$.*

1. *M is sublocally constant if and only if M is isomorphic to a subfunctor of a constant functor.*
2. *If M is a sublocally constant representation then for every morphism $\alpha : x \to y$ in $\mathcal{C}$, $M(\alpha)$ is an inclusion $M(x) \hookrightarrow M(y)$.*

(3) *If M is a subfunctor of a constant functor $\underline{A}$ for some group A then $\varinjlim M$ is the subgroup of A generated by the $M(x)$ where x ranges over the objects of $\mathcal{C}$.*

Proof. A subfunctor of a constant functor is also a subfunctor of a locally constant functor. Conversely, given a subfunctor of a locally constant functor, it is isomorphic to a subfunctor of the constant functor whose group is the direct sum over the connected components of $\mathcal{C}$ of the component groups of the locally constant functor. This proves (1)

The second statement holds because $M(\alpha)$ is the restriction of an identity morphism.

For the third statement we observe that the subgroup generated by the $M(x)$ satisfies the universal property of the direct limit. □

Given any representation M we now construct

- a constant functor M^{c},
- a locally constant functor M^{lc},
- a sublocally constant functor M^{slc}

together with morphisms

$$M \to M^{\mathrm{slc}} \hookrightarrow M^{\mathrm{lc}} \hookrightarrow M^{\mathrm{c}}$$

which turn out to have a certain universal property. We define $M^{\mathrm{c}}(x) = \varinjlim M$ for all x in $\mathcal{C}$, so that

$$M^{\mathrm{c}} = \underline{\varinjlim M}$$

is the constant functor with value $\varinjlim M$. For each connected component $\mathcal{D}$ of $\mathcal{C}$ we define $M^{\mathrm{lc}}(x) = \varinjlim(M\downarrow_{\mathcal{D}})$ for each $x \in \mathcal{D}$, where $M\downarrow_{\mathcal{D}}$ denotes the restriction of M to $\mathcal{D}$ and the limit is taken over $\mathcal{D}$ (not $\mathcal{C}$). For each morphism $\alpha : x \to y$, $M^{\mathrm{lc}}(\alpha)$ is defined to be the identity morphism. Finally we define $M^{\mathrm{slc}}(x)$ to be the image of the canonical homomorphism $M(x) \to \varinjlim M$, and we define the effect of morphisms on M^{slc} so that it becomes a subfunctor of M^{c}. This works because if $\alpha : x \to y$ is a morphism in $\mathcal{C}$ then the homomorphism $M(x) \to \varinjlim M$ factors as $M(x) \to M(y) \to \varinjlim M$ and so the image of $M(x)$ in $\varinjlim M$ is contained in the image of $M(y)$. The morphism $M^{\mathrm{slc}}(\alpha)$ is the inclusion map. From the construction it is clear that $M^{\mathrm{slc}} \subseteq M^{\mathrm{lc}} \subseteq M^{\mathrm{c}}$. The map $M \to M^{\mathrm{slc}}$ is the map specified at each object x as the canonical map from $M(x)$ to its image in $\varinjlim M$, and it is a surjection.

We characterize sublocally constant representations in various ways.

Proposition 7.2. *The following are equivalent for a representation M.*

(1) *M is sublocally constant,*
(2) *the canonical quotient map $M \to M^{\mathrm{slc}}$ is an isomorphism,*
(3) *each of the canonical homomorphisms $M(x) \to \varinjlim M$, as x ranges over the objects of $\mathcal{C}$, is a monomorphism,*
(4) *M is isomorphic to a subfunctor of the constant functor M^{c}.*
(5) *M is isomorphic to a subfunctor of the locally constant functor M^{lc}.*

Proof. All of (2), (4) and (5) immediately imply (1) from the definitions, and (2) is equivalent to (3) from the definitions. Since M^{slc} is a subfunctor of both M^{lc} and M^{c}, (2) implies both (4) and (5). We see that (1) implies (3) because if M is sublocally constant we may regard it as a subfunctor of a constant functor $\underline{A}$ and by Lemma 7.1 each $M(x)$ is a subgroup of $\varinjlim M$. □

We now state the universal property which these constructions satisfy.

Proposition 7.3. *For every representation M there is a constant representation M^{c}, a locally constant representation M^{lc} and a sublocally constant representation M^{slc} together with morphisms*

$$M \to M^{\mathrm{slc}} \hookrightarrow M^{\mathrm{lc}} \hookrightarrow M^{\mathrm{c}}$$

the first being a surjection and the second and third inclusions, with the property that

- *every morphism $M \to N$ where N is constant factors uniquely as $M \to M^{\mathrm{c}} \to N$,*
- *every morphism $M \to N$ where N is locally constant factors uniquely as $M \to M^{\mathrm{lc}} \to N$ and*
- *every morphism $M \to N$ where N is sublocally constant factors uniquely as $M \to M^{\mathrm{slc}} \to N$.*

Furthermore, both of the maps $M \to M^{\mathrm{slc}} \to M^{\mathrm{lc}}$ induce isomorphisms $\varinjlim M \to \varinjlim M^{\mathrm{slc}} \to \varinjlim M^{\mathrm{lc}}$.

Proof. Let N be a sublocally constant representation and suppose we are given a morphism $M \to N$. We will show that this morphism factors through $M \to M^{\mathrm{slc}}$. For each object x of $\mathcal{C}$ we have

$$\mathrm{Ker}(M(x) \to \varinjlim M) \subseteq \mathrm{Ker}(M(x) \to \varinjlim N) = \mathrm{Ker}(M(x) \to N(x))$$

since by Proposition 7.2 part (3) the canonical homomorphism $N(x) \to \varinjlim N$ is an injection. Hence each morphism $M(x) \to N(x)$ factors through $M(x) \to M^{\mathrm{slc}}(x)$ and this shows that $M \to N$ factors through $M \to M^{\mathrm{slc}}$, the uniqueness of factorization being clear.

If now we have a morphism $M \to N$ where N is either constant or locally constant, then in particular N is sublocally constant and we have a factorization through M^{slc}. We readily see that the mapping $M^{\mathrm{slc}} \to N$ extends to a mapping defined on M^{lc} or M^{c}, as appropriate and again uniqueness of factorization is clear.

Since M^{lc} is constant on each component and the direct sum of the component groups is $\varinjlim M$ we have an isomorphism $\varinjlim M^{\mathrm{lc}} \to \varinjlim M$ such that the composite $\varinjlim M \to \varinjlim M^{\mathrm{slc}} \to \varinjlim M^{\mathrm{lc}} \to \varinjlim M$ is the identity. Observe that $\varinjlim M \to \varinjlim M^{\mathrm{slc}}$ is surjective since $\varinjlim$ is right exact. The final assertion about direct limits follows. $\square$

A more sophisticated approach to the statements in Lemma 7.3 is to say that each of the functors $M \to M^{\mathrm{c}}$, $M \to M^{\mathrm{lc}}$ and $M \to M^{\mathrm{slc}}$ is left adjoint to the corresponding functor which includes either constant, locally constant or sublocally constant functors in the full category of representations of $\mathcal{C}$. The morphisms at the start of Proposition 7.3 provide the units of these adjunctions.

8. Stem Extensions of Categories

We define a *stem extension* of categories to be an extension $(K|\mathcal{E})$ which satisfies any of the equivalent statements in the next proposition. Some of these statements have to do with the equivalent $\mathbb{Z}\mathcal{C}$-module extension $(K|\mathcal{E}^\dagger)$ and we will say also that this module extension is a stem extension in this case.

Proposition 8.1. *Let $(K|\mathcal{E}) = (K \to \mathcal{E} \to \mathcal{C})$ be an extension of $\mathcal{C}$ by a $\mathbb{Z}\mathcal{C}$-module K with corresponding $\mathbb{Z}\mathcal{C}$-module extension $(K|\mathcal{E}^\dagger)$ of ${}^\bullet I\mathcal{C}$. The following statements are equivalent.*

1. *The induced map $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism.*
2. *The induced map $\varinjlim \mathcal{E}^\dagger \to \varinjlim {}^\bullet I\mathcal{C}$ is an isomorphism.*
3. *The induced map $\varinjlim K \to \varinjlim \mathcal{E}^\dagger$ is zero.*
4. *The induced map $H_1(K) \to H_1(\mathcal{E})$ is zero.*
5. *$K \subseteq I\mathcal{C}^\bullet \cdot \mathcal{E}^\dagger$.*

Proof. (1) $\Leftrightarrow$ (5) We have $H_1(\mathcal{E}) \cong (I\mathcal{E}^\bullet \cap {}^\bullet I\mathcal{E})/(I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E})$ by Corollary 2.4 and $H_1(\mathcal{C}) \cong (I\mathcal{E}^\bullet \cap {}^\bullet I\mathcal{E})/(N + I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E})$ where $N = \mathrm{Ker}(\mathbb{Z}\mathcal{E} \to$

$\mathbb{Z}\mathcal{C}$). The map $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is the quotient map. It is an isomorphism if and only if $N + I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E} = I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E}$, if and only if $N \subseteq I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E}$. Now this happens if and only if

$$K = N/(N \cdot {}^\bullet I\mathcal{E}) \subseteq (I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E})/(N \cdot {}^\bullet I\mathcal{E}) = I\mathcal{E}^\bullet \cdot ({}^\bullet I\mathcal{E}/(N \cdot {}^\bullet I\mathcal{E})) = I\mathcal{C}^\bullet \cdot \mathcal{E}^\dagger.$$

(5) $\Leftrightarrow$ (3) The map in (3) is $K/(I\mathcal{C}^\bullet \cdot K) \to \mathcal{E}^\dagger/(I\mathcal{C}^\bullet \cdot \mathcal{E}^\dagger)$ and this is zero if and only if $K \subseteq I\mathcal{C}^\bullet \cdot \mathcal{E}^\dagger$.

(2) $\Leftrightarrow$ (3) Applying $\varinjlim$ to the short exact sequence $0 \to K \to \mathcal{E}^\dagger \to {}^\bullet I\mathcal{C} \to 0$ gives an exact sequence $\varinjlim K \to \varinjlim \mathcal{E}^\dagger \to \varinjlim {}^\bullet I\mathcal{C} \to 0$ and the equivalence of (2) and (3) is now clear.

(1) $\Leftrightarrow$ (4) We have $H_1(K) = K$ since the groups in K are abelian, and the map $H_1(K) \to H_1(\mathcal{E})$ is

$$K \cong \frac{N}{N \cdot {}^\bullet I\mathcal{E}} \to \frac{I\mathcal{E}^\bullet \cap {}^\bullet I\mathcal{E}}{I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E}}$$

induced by the inclusion of N in $I\mathcal{E}^\bullet \cap {}^\bullet I\mathcal{E}$. This map is zero if and only if $N \subseteq I\mathcal{E}^\bullet \cdot {}^\bullet I\mathcal{E}$, which was an equivalent condition to (1) which appeared in the proof of the equivalence of (1) and (5). □

The special case of the next result which arises when the extensions are group extensions is a standard result and has a direct proof using a calculation with commutators of group elements. Such an approach is not available to us here, but the result is immediate from the five-term exact sequence.

Proposition 8.2. *In any map of extensions*

$$\begin{array}{ccccc} K_1 & \to & \mathcal{E}_1 & \to & \mathcal{C} \\ \downarrow & & \downarrow & & \| \\ K & \to & \mathcal{E} & \to & \mathcal{C} \end{array}$$

where K_1 and K are locally constant and the lower extension is a stem extension, the homomorphism $K_1 \to K$ is surjective.

Proof. Consider the five-term exact sequences

$$\begin{array}{ccccccccc} H_2(\mathcal{E}_1) & \to & H_2(\mathcal{C}) & \to & \varinjlim K_1 & \to & H_1(\mathcal{E}_1) & \to & H_1(\mathcal{C}) \\ \downarrow & & \| & & \downarrow & & \downarrow & & \| \\ H_2(\mathcal{E}) & \to & H_2(\mathcal{C}) & \to & \varinjlim K & \to & H_1(\mathcal{E}) & \to & H_1(\mathcal{C}) \end{array}.$$

Since $H_1(\mathcal{E}) \to H_1(\mathcal{C})$ is an isomorphism $H_2(\mathcal{C}) \to \varinjlim K$ is surjective, and since this factors through $\varinjlim K_1$ the map $\varinjlim K_1 \to \varinjlim K$ must be surjective also. Since K_1 and K are locally constant it follows that $K_1 \to K$ is surjective. □

We may put a transitive relation $\geq$ on the set of stem extensions of $\mathcal{C}$ by locally constant functors by writing $(K_1|\mathcal{E}_1) \geq (K|\mathcal{E})$ if there is a morphism $(K_1|\mathcal{E}_1) \to (K|\mathcal{E})$. We see from Theorem 6.1 and Proposition 8.2 that provided $H_2(\mathcal{C})$ is a finitely generated abelian group, if $(K_1|\mathcal{E}_1) \geq (K|\mathcal{E})$ and $(K|\mathcal{E}) \geq (K_1|\mathcal{E}_1)$ then $(K|\mathcal{E}) \cong (K_1|\mathcal{E}_1)$ since the two composites of the two morphisms when restriced to K and K_1 are both surjections, and hence isomorphisms. Thus $\geq$ induces a partial order on the set of isomorphism classes of stem extensions. We say that a stem extension $(K|\mathcal{E})$ of $\mathcal{C}$ by a locally constant functor is *maximal* if its isomorphism class is maximal in this partial order. Equivalently, a stem extension by a locally constant functor is maximal if and only if whenever it is the target of a morphism from a stem extension, that morphism is an isomorphism. We have as our goals Theorems 8.4 and 8.6 which are about the maximal stem extensions.

Lemma 8.3. *Let M be a sublocally constant representation and let $M \to M^{\rm lc}$ be the canonical embedding in a locally constant functor. Given an extension $(M|\mathcal{E}) : M \to \mathcal{E} \to \mathcal{C}$ let the extension $(M^{\rm lc}|\tilde{\mathcal{E}}) : M^{\rm lc} \to \tilde{\mathcal{E}} \to \mathcal{C}$ be obtained by explicit pushout.*

1. *If $(M|\mathcal{E})$ is a stem extension then so is $(M^{\rm lc}|\tilde{\mathcal{E}})$.*
2. *Any morphism of extensions from $(M|\mathcal{E})$ to an extension with locally constant kernel factors through the morphism $(M|\mathcal{E}) \to (M^{\rm lc}|\tilde{\mathcal{E}})$.*

Proof. (1) Consider the five-term exact sequences

$$\begin{array}{ccccccccc} H_2(\mathcal{E}) & \to & H_2(\mathcal{C}) & \to & \varinjlim M & \to & H_1(\mathcal{E}) & \to & H_1(\mathcal{C}) \\ \downarrow & & \| & & \| & & \downarrow & & \| \\ H_2(\tilde{\mathcal{E}}) & \to & H_2(\mathcal{C}) & \to & \varinjlim M^{\rm lc} & \to & H_1(\tilde{\mathcal{E}}) & \to & H_1(\mathcal{C}) \end{array} .$$

Here the top map $H_2(\mathcal{C}) \to \varinjlim M$ is epi and hence so is the map $H_2(\mathcal{C}) \to \varinjlim M^{\rm lc}$. Thus $H_1(\tilde{\mathcal{E}}) \to H_1(\mathcal{C})$ is an isomorphism and $(M^{\rm lc}|\tilde{\mathcal{E}})$ is a stem extension.

(2) Suppose we have a map of extensions $(M|\mathcal{E}) \to (L|\mathcal{G})$ where L is locally constant. The map $M \to L$ factors as $M \xrightarrow{\alpha} M^{\rm lc} \xrightarrow{\beta} L$ by Proposition 7.3 and if the second cohomology classes of the three extensions $(M|\mathcal{E}), (M^{\rm lc}|\tilde{\mathcal{E}})$ and $(L|\mathcal{G})$ are $\epsilon, \tilde{\epsilon}$ and γ then $\gamma = (\beta\alpha)_*(\epsilon) = \beta_*(\alpha_*(\epsilon))$ and $\alpha_*(\epsilon) = \tilde{\epsilon}$. This implies there is a morphism of extensions $(M^{\rm lc}|\tilde{\mathcal{E}}) \to (L|\mathcal{G})$ as required, by Proposition 3.1. □

The next result identifies the maximal stem extensions by locally constant functors, and is one of the principal results of this section. It says that each maximal stem extension is, on each component of $\mathcal{C}$, an extension by the constant functor whose group is the Schur multiplier of that component. This is a functor which, when $\mathcal{C}$ is not connected, will usually differ from the constant functor $\underline{H_2(\mathcal{C})}$, but which in all cases has $H_2(\mathcal{C})$ as its direct limit. To simplify the question of dealing with connected components of $\mathcal{C}$ we assume that $\mathcal{C}$ is connected in the next result, so as to obtain a cleaner statement.

Theorem 8.4. *Let $\mathcal{C}$ be a connected category and let $(K|\mathcal{E}) : K \to \mathcal{E} \to \mathcal{C}$ be a stem extension, where K is a constant representation of $\mathcal{C}$. Then $(K|\mathcal{E})$ is an image of a stem extension $(\underline{H_2(\mathcal{C})}|\mathcal{D}) : \underline{H_2(\mathcal{C})} \to \mathcal{D} \to \mathcal{C}$.*

Proof. Let $\mathcal{F} \to \mathcal{E}$ be a surjective functor where $\mathcal{F}$ is a free category with the same objects as $\mathcal{C}$, so that we obtain by Theorem 3.7 a morphism of extensions

$$\begin{array}{ccccc} B & \to & \mathcal{F}^{\dagger\dagger} & \to & \mathcal{C} \\ \downarrow & & \downarrow & & \| \\ K & \to & \mathcal{E} & \to & \mathcal{C} \end{array}$$

where $B = N/(N \cdot {}^\bullet I\mathcal{F})$ is the relation module of the presentation $\mathcal{F} \to \mathcal{E} \to \mathcal{C}$ and N is the kernel of the algebra homomorphism $\mathbb{Z}\mathcal{F} \to \mathbb{Z}\mathcal{C}$. It is convenient to work with the equivalent diagram of $\mathbb{Z}\mathcal{C}$-modules

$$\begin{array}{ccccc} B & \to & \mathcal{F}^{\dagger} & \to & {}^\bullet I\mathcal{C} \\ \downarrow & & \downarrow & & \| \\ K & \to & \mathcal{E}^{\dagger} & \to & {}^\bullet I\mathcal{C} \end{array}$$

where $\mathcal{F}^\dagger = {}^\bullet I\mathcal{F}/(N \cdot {}^\bullet I\mathcal{F})$, using the equivalence of Theorem 3.6. We have an isomorphism,

$$H_2(\mathcal{C}) = \frac{(I\mathcal{F}^\bullet \cdot {}^\bullet I\mathcal{F}) \cap N}{I\mathcal{F}^\bullet \cdot N + N \cdot {}^\bullet I\mathcal{F}} \cong \frac{(I\mathcal{C}^\bullet \cdot \mathcal{F}^\dagger) \cap B}{I\mathcal{C}^\bullet \cdot B}$$

by Corollary 2.4, and this is a subgroup of $\varinjlim B = B/(I\mathcal{C}^\bullet \cdot B)$.

We have seen in Theorem 6.1 that the composite map $H_2(\mathcal{C}) \to \varinjlim B \to \varinjlim K$ is surjective, so that if $Y = \operatorname{Ker}(\varinjlim B \to \varinjlim K)$ then $Y + H_2(\mathcal{C}) = \varinjlim B$. We claim that there is in fact a direct sum decomposition $\varinjlim B = X \oplus H_2(\mathcal{C})$ where X is a subgroup contained in Y which hence maps to 0 in $\varinjlim K$. This is because the quotient

$$\frac{\varinjlim B}{H_2(\mathcal{C})} \cong \frac{B}{I\mathcal{C}^\bullet \mathcal{F}^\dagger \cap B} \cong \frac{(I\mathcal{C}^\bullet \cdot \mathcal{F}^\dagger) + B}{I\mathcal{C}^\bullet \cdot \mathcal{F}^\dagger}$$

is isomorphic to a subgroup of $\mathcal{F}^\dagger/(I\mathcal{C}^\bullet \cdot \mathcal{F}^\dagger) = \varinjlim \mathcal{F}^\dagger \cong \varinjlim {}^\bullet I\mathcal{F}$ and ${}^\bullet I\mathcal{F}$ is a projective $\mathbb{Z}\mathcal{F}$-module, which is in fact isomorphic to a direct sum of representable functors F_x by Proposition 2.2. We see that $\varinjlim {}^\bullet I\mathcal{F}$ is a free abelian group, since by an elementary calculation $\varinjlim F_x = \mathbb{Z}$ for each object x of $\mathcal{C}$. It follows that $(\varinjlim B)/H_2(\mathcal{C}) \cong Y/(Y \cap H_2(\mathcal{C}))$ is a free abelian group and we conclude that $Y = X \oplus (Y \cap H_2(\mathcal{C}))$ for some direct factor $X \subseteq Y$, and hence that $\varinjlim B = X \oplus H_2(\mathcal{C})$.

Let $p : \varinjlim B \to H_2(\mathcal{C})$ be the projection onto $H_2(\mathcal{C})$ defined by the decomposition $\varinjlim B = X \oplus H_2(\mathcal{C})$. We construct a sublocally constant representation U of $\mathcal{C}$ by defining for each object x of $\mathcal{C}$ the value $U(x) = \mathrm{Im}(B(x) \to \varinjlim B \xrightarrow{p} H_2(\mathcal{C}))$. The natural map $B \to U$ specified at each x by $B(x) \to \varinjlim B \to U(x)$ is surjective by construction. Furthermore $\varinjlim U = H_2(\mathcal{C})$ since p is surjective. Let the extension $(U|J)$ be defined by explicit pushout along this map, so there is a morphism of extensions $(B|\mathcal{F}^\dagger) \to (U|J)$.

We claim that $(U|J)$ is a stem extension, and we prove it by showing that the map $\varinjlim U \to \varinjlim J$ is zero, one of the equivalent conditions of Proposition 8.1. We apply the functor $\varinjlim$ to the left hand square of the morphism of extensions $(B|\mathcal{F}^\dagger) \to (U|J)$ to obtain a commutative diagram of groups

$$\begin{array}{ccc} \varinjlim B & \longrightarrow & \varinjlim \mathcal{F}^\dagger \\ \downarrow & & \downarrow \\ \varinjlim U & \longrightarrow & \varinjlim J \end{array} .$$

Observe that $H_2(\mathcal{C})$ is the kernel of the top map so that the image of $\varinjlim B$ in $\varinjlim \mathcal{F}^\dagger$ equals the image of X in $\varinjlim \mathcal{F}^\dagger$. Also $\varinjlim B \to \varinjlim U$ is surjective (since $\varinjlim$ is right exact) so that the image of $\varinjlim U$ in $\varinjlim J$ is

$$\begin{aligned} \mathrm{Im}(\varinjlim U \to \varinjlim J) &= \mathrm{Im}(\varinjlim B \to \varinjlim U \to \varinjlim J) \\ &= \mathrm{Im}(\varinjlim B \to \varinjlim \mathcal{F}^\dagger \to \varinjlim J) \\ &= \mathrm{Im}(X \to \varinjlim \mathcal{F}^\dagger \to \varinjlim J) \\ &= \mathrm{Im}(X \to \varinjlim U \to \varinjlim J) \\ &= 0 \end{aligned}$$

since X was constructed to lie in $\operatorname{Ker}(\varinjlim B \to \varinjlim U)$. This shows that $(U|J)$ is a stem extension.

We claim that because $X \subseteq \operatorname{Ker}(\varinjlim B \to \varinjlim K)$ and K is locally constant the morphism of extensions $(B|\mathcal{F}^\dagger) \to (K|\mathcal{E}^\dagger)$ factors through $(U|J)$. This is because $U \cong B/Q$ where

$$Q(x) = \operatorname{Ker}(B(x) \to \varinjlim B \xrightarrow{p} H_2(\mathcal{C})).$$

Now

$$\begin{aligned}\operatorname{Ker}(Q(x) \to K(x)) &= \operatorname{Ker}(Q(x) \to \varinjlim K)\\ &= \operatorname{Ker}(Q(x) \to \varinjlim B \to \varinjlim K)\\ &= Q(x)\end{aligned}$$

since

$$\begin{aligned}Q(x) &= \operatorname{Ker}(B(x) \to \varinjlim B \xrightarrow{p} H_2(\mathcal{C}))\\ &= \operatorname{Ker}(B(x) \to \operatorname{Ker} p)\\ &= \{z \in B(x) \bigm| \operatorname{Im}(z) \in X\}\end{aligned}$$

and this implies that $Q(x)$ is sent to 0 in $\varinjlim K$. Hence $Q(x)$ is sent to 0 in K, since $X \subseteq Y = \operatorname{Ker}(\varinjlim B \to \varinjlim K)$.

Next the morphism $(U|J) \to (K|\mathcal{E}^\dagger)$ factors through the morphism $(U|J) \to (\underline{H_2(\mathcal{C})}|J)$ which we see by recalling that $\varinjlim U = H_2(\mathcal{C})$. By Lemma 8.3 the extension $(\underline{H_2(\mathcal{C})}|J)$ is stem. □

Corollary 8.5. *Let $\mathcal{C}$ be a category for which $H_2(\mathcal{C})$ is finitely generated. The maximal stem extensions of $\mathcal{C}$ by locally constant functors are precisely the stem extensions which on each component $\mathcal{D}$ of $\mathcal{C}$ have the form $(\underline{H_2(\mathcal{D})}|\mathcal{E})$.*

Proof. First we comment that it suffices to assume that $\mathcal{C}$ is connected, because in general an extension of $\mathcal{C}$ is a disjoint union of extensions of its connected components, and the extension is stem if and only if on each component the extension is stem. Thus we assume $\mathcal{C} = \mathcal{D}$ is connected.

We see from Theorem 8.4 that every maximal stem extension has the form stated.

Conversely, suppose we have a stem extension $(\underline{H_2(\mathcal{C})}|\mathcal{E})$. Given a morphism $(K|\mathcal{E}_1) \to (\underline{H_2(\mathcal{C})}|\mathcal{E})$ in which the first extension is also stem, by Theorem 8.4 there is a morphism $(\underline{H_2(\mathcal{C})}|\mathcal{E}_2) \to (K|\mathcal{E}_1)$ for some $\mathcal{E}_2$, giving a composite morphism $(\underline{H_2(\mathcal{C})}|\mathcal{E}_2) \to (\underline{H_2(\mathcal{C})}|\mathcal{E})$. By Proposition 8.2 all these morphisms must be epimorphisms, and in particular the epimorphism $\underline{H_2(\mathcal{C})} \to \underline{H_2(\mathcal{C})}$ is a surjection. Since $H_2(\mathcal{C})$ is finitely generated this epimorphism is an isomorphism and hence the original morphisms $K \to \underline{H_2(\mathcal{C})}$ and $(K|\mathcal{E}_1) \to (\underline{H_2(\mathcal{C})}|\mathcal{E})$ are isomorphisms. Hence $(\underline{H_2(\mathcal{C})}|\mathcal{E})$ is a maximal stem extension. □

We have already proved in Theorem 6.2 that if $H_1(\mathcal{C})$ is a free abelian group and $H_2(\mathcal{C})$ is finitely generated then up to isomorphism there is a extension by $\underline{H_2(\mathcal{C})}$ with the property that every extension of $\mathcal{C}$ by a constant functor can be obtained from it by explicit pushout, but we did not prove that this is a stem extension (assuming $\mathcal{C}$ is connected). We now do this.

Theorem 8.6. *Let $\mathcal{C}$ be a connected category for which $H_1(\mathcal{C})$ is free abelian and $H_2(\mathcal{C})$ is finitely generated. There is up to isomorphism a unique maximal stem extension of $\mathcal{C}$ by a constant functor. It has the form $(\underline{H_2(\mathcal{C})}|\mathcal{E})$ and has the property that every extension of $\mathcal{C}$ by a constant functor can be obtained from it by explicit pushout.*

Proof. Let $(\underline{H_2(\mathcal{C})}|\mathcal{E})$ be the extension constructed in Theorem 6.2, so that every extension by a constant functor can be obtained from it by explicit pushout. Let $(\underline{H_2(\mathcal{C})}|\mathcal{E}_1)$ be any maximal stem extension of $\mathcal{C}$ by a constant functor. It can be obtained by explicit pushout from the first extension, so there is a morphism $(\underline{H_2(\mathcal{C})}|\mathcal{E}) \to (\underline{H_2(\mathcal{C})}|\mathcal{E}_1)$. By Proposition 8.2 this morphism must be an epimorphism, and since $H_2(\mathcal{C})$ is finitely generated the surjection $H_2(\mathcal{C}) \to H_2(\mathcal{C})$ must be an isomorphism. Hence the two extensions are isomorphic, and the extension described in Theorem 6.2 is a stem extension. □

9. Examples

9.1. **Cyclic groups.** Let $H = H(s)$ be the free monoid on a single generator s and let $C_m = \langle x \rangle$ be a cyclic group of order m. The surjective homomorphism $H \to C_m$ which sends s to x gives rise to a surjection $RH \to RC_m$, and we let N be the kernel. We see that the monoid algebra RH is a polynomial ring $R[s]$, the augmentation ideal IH is the ideal $(s-1)$ and $N = (s^m - 1)$. The resolution described in Theorem 2.3 is now

$$\cdots \to \frac{(s^m-1)^2}{(s^m-1)^3} \to \frac{(s^m-1)(s-1)}{(s^m-1)^2(s-1)}$$
$$\to \frac{(s^m-1)}{(s^m-1)^2} \to \frac{(s-1)}{(s^m-1)(s-1)} \to \frac{R[s]}{(s^m-1)} \to R \to 0$$

and this identifies with the usual resolution of R over RC_m since as representations of RC_m each term (apart from R) is a copy of the regular representation, and we can identify mappings such as

$$\frac{(s^m - 1)}{(s^m - 1)^2} \to \frac{(s - 1)}{(s^m - 1)(s - 1)}$$

as sending the generator of the first module to $1 + s + \cdots + s^{m-1}$ times the generator of the second module and mappings such as

$$\frac{(s^m - 1)(s - 1)}{(s^m - 1)^2(s - 1)} \to \frac{(s^m - 1)}{(s^m - 1)^2}$$

as sending the generator of the first module to $s-1$ times the generator of the second module.

9.2. **The suspension of $K(C_2, 1)$.** Let $\mathcal{C}$ be the category with three objects labelled x, y and z and three non-identity morphisms: $g : x \to x$, $a : x \to y$ and $b : x \to z$ and so that $g^2 = 1_x$, the other compositions being determined uniquely. The nerve of $\mathcal{C}$ is the suspension of the Eilenberg-MacLane space $K(C_2, 1)$, since the endomorphism monoid of x has as its nerve $K(C_2, 1)$, adjoining just one of the morphisms a or b produces a cone on this space, and adjoining both a and b gives the double cone, or suspension. Thus the homology of $\mathcal{C}$ is

$$H_n(\mathcal{C}) = \begin{cases} \mathbb{Z} & \text{if } r = 0, \\ 0 & \text{if } n \text{ is odd}, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

We take the presentation of $\mathcal{C}$ by the free category $\mathcal{F}$ with objects x, y and z and generator morphisms $G : x \to x$, $A : x \to y$ and $B : x \to z$. The left augmentation ideal of $\mathcal{F}$ is

$$^{\bullet}I\mathcal{F} = \mathbb{Z}\mathcal{F}(1_x - G) + \mathbb{Z}\mathcal{F}(1_y - A) + \mathbb{Z}\mathcal{F}(1_z - B)$$

and the kernel N of the algebra homomorphism $\mathbb{Z}\mathcal{F} \to \mathbb{Z}\mathcal{C}$ is the 2-sided ideal generated by $1_x - G^2$, $B(1_x - G)$ and $A(1_x - G)$. We find that

$$N \cdot {^{\bullet}I\mathcal{F}} = \mathbb{Z}\mathcal{F}(1_x - G^2)(1_x - G) + \mathbb{Z}A(1_x - G)^2 + \mathbb{Z}B(1_x - G)^2$$

noting that many products of terms vanish in this computation. Making use of the identity $1_x - G^2 = -(1_x - G)^2 + 2(1_x - G)$ we find that, modulo $N \cdot {^{\bullet}I\mathcal{F}}$, N is spanned by $1_x - G^2$, $A(1_x - G)$ and $B(1_x - G)$. The images of these elements in $N/(N \cdot {^{\bullet}I\mathcal{F}})$ are independent (on considering the effects of left multiplication by 1_x, 1_y and 1_z) and we have $A(1 - G^2) \equiv 2A(1 - G)$ and $B(1 - G^2) \equiv 2B(1 - G)$. Thus the relation module M associated to this presentation has $M(x) = M(y) = M(z) = \mathbb{Z}$ and $M(G)$ acts as the identity on $M(x)$, $M(A)$ includes $M(x)$ into $M(y)$ as $\mathbb{Z} \to 2\mathbb{Z}$ and also $M(B)$ includes $M(x)$ into $M(z)$ as $\mathbb{Z} \to 2\mathbb{Z}$.

We see from this that the projective extension of $\mathcal{C}$ described in Theorem 3.7 has as its middle term the category $\mathcal{F}^{\dagger\dagger}$ which has objects x, y and z and with each of the morphism sets $\mathrm{End}(x)$, $\mathrm{End}(y)$, $\mathrm{End}(z)$, $\mathrm{Hom}(x,y)$ and $\mathrm{Hom}(x,z)$ a copy of $\mathbb{Z}$. The composition of any two composable morphisms is the sum of the integers. The surjection $\mathcal{F}^{\dagger\dagger} \to \mathcal{C}$ sends each of these sets of morphisms to a single morphism, except for $\mathrm{End}_{\mathcal{F}}(x) \to \mathrm{End}_{\mathcal{C}}(x)$ which is a surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. This surjection of categories is seen to be part of an extension

$$M \to \mathcal{F}^{\dagger\dagger} \to \mathcal{C}.$$

Since $H_1(\mathcal{C}) = 0$ and $H_2(\mathcal{C}) = \mathbb{Z}/2\mathbb{Z}$ there is a unique (up to isomorphism) maximal stem extension of $\mathcal{C}$ by a locally constant functor, and its left term is the constant functor $\underline{\mathbb{Z}/2\mathbb{Z}}$. We describe it first and then explain how it may be calculated. The extension category $\mathcal{E}$ has objects x, y and z and now $\mathrm{End}(y)$ and $\mathrm{End}(z)$ are copies of the cyclic group C_2 while $\mathrm{End}(x)$ is a Klein four-group $C_2 \times C_2$. We may identify $\mathrm{Hom}(x,y)$ as a copy of C_2 in which $\mathrm{End}(x)$ acts via composition as projection onto the first factor $p_1 : C_2 \times C_2 \to C_2$ and then multiplication within C_2, and $\mathrm{End}(y)$ acts as multiplication within C_2. On the other hand $\mathrm{Hom}(x,z)$ is a copy of C_2 in which $\mathrm{End}(x)$ acts via composition as projection onto the second factor $p_2 : C_2 \times C_2 \to C_2$ and then multiplication within C_2, and $\mathrm{End}(z)$ acts as multiplication within C_2. There is an extension

$$\underline{\mathbb{Z}/2\mathbb{Z}} \to \mathcal{E} \to \mathcal{C}$$

in which the surjection $\mathrm{End}_{\mathcal{E}}(x) \to \mathrm{End}_{\mathcal{C}}(x)$ is the homomorphism $C_2 \times C_2 \to C_2$ which has the diagonally embedded copy of C_2 as its kernel, and this is the maximal stem extension.

We may compute this extension by first working over the field of two elements $\mathbb{F}_2$ and constructing a projective resolution of the constant functor $\underline{\mathbb{F}_2}$ by projective $\mathbb{F}_2\mathcal{C}$-modules. We write modules diagrammatically, so that the constant functor $\underline{\mathbb{F}_2}$ has the form

$$\underline{\mathbb{F}_2} = \begin{matrix} & S_x & \\ S_y & & S_z \end{matrix}$$

where we denote the three simple $\mathbb{F}_2\mathcal{C}$-modules, each of dimension 1 on the subscript object and 0 on the other two, by S_x, S_y and S_z. we have a decomposition of the regular representation into indecomposable projective modules

$$\mathbb{F}_2\mathcal{C} = P_x \oplus P_y \oplus P_z = \begin{matrix} & S_x & \\ S_x & S_y & S_z \end{matrix} \oplus S_y \oplus S_z$$

and the start of a resolution

$$S_x \oplus S_y \oplus S_z \to P_x \to P_x \to \begin{matrix} & S_x & \\ S_y & & S_z \end{matrix} \to 0.$$

Using this we may compute $H^2(\mathcal{C}, \underline{\mathbb{F}_2})$ by the exact sequence

$$\mathrm{Hom}(P_x, \underline{\mathbb{F}_2}) \to \mathrm{Hom}(S_x \oplus S_y \oplus S_z, \underline{\mathbb{F}_2}) \to H^2(\mathcal{C}, \underline{\mathbb{F}_2}) \to 0$$

and we see that $H^2(\mathcal{C}, \underline{\mathbb{F}_2})$ has order 2 with a non-zero element in cohomology represented by the morphism $S_x \oplus S_y \oplus S_z \to \begin{smallmatrix} & S_x & \\ S_y & & S_z \end{smallmatrix} \to 0$ which is non-zero on the S_y summand and zero on the other two summands. Translating this to the relation module which we have already constructed, the stem extension we require is the explicit pushout of the projective extension $M \to \mathcal{F}^{\dagger\dagger} \to \mathcal{C}$ along the morphism $M \to \underline{\mathbb{F}_2}$ given as surjection $\mathbb{Z} \to \mathbb{F}_2$ at the object y and zero at the other objects. Following the definition of the explicit pushout now gives the desired stem extension.

9.3. **Fusion systems and p-local finite groups.** We refer to [2] for background on these topics. In [16] Linckelmann defines the Schur multiplier of a fusion system on a finite p-group P to be the elements of $H^2(P, \mathbb{C}^\times)$ which are stable under the operations in the fusion system, or equivalently the elements of $H_2(P, \mathbb{Z})$ which are thus stable. When the fusion system arises from a finite group this construction gives the usual Schur multiplier, by the stable elements theorem of Cartan and Eilenberg [5]. In general when the fusion system is part of a p-local finite group, with a linking system $\mathcal{L}$, it is possible that Linckelmann's definition may coincide with $H_2(\mathcal{L}, \mathbb{Z})$, which in our terminology is the Schur multiplier of the linking system. However this seems not to be proven in the literature. It is stated as Theorem 4.7 of [2] that (co)homology of the linking system coincides with (co)homology stable under the fusion system when mod p coefficients are taken, but not when integral coefficients are taken.

In this area of mathematics, we comment also that extensions of categories by constant functors play a role in the discussion of central extensions of p-local finite groups in [3, Sect. 6]. Such central extensions are defined there, and it is observed that they correspond to extensions of the associated quasicentric linking system by constant functors.

References

[1] H.-J. Baues and G. Wirsching, *Cohomology of small categories*, J. Pure Appl. Algebra **38** (1985), 187–211.

[2] C. Broto, R. Levi and B. Oliver, *The theory of p-local groups: a survey.* Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, 51–84, Contemp. Math. **346**, Amer. Math. Soc., Providence, RI, 2004.

[3] C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver, *Extensions of p-local finite groups*, Trans. Amer. Math. Soc. **359** (2007), 3791-3858.

[4] M.C.R. Butler and A. King, *Minimal resolutions of algebras*, J. Algebra **212** (1999), 323–362.

[5] H. Cartan and S. Eilenberg, *Homological Algebra,* Princeton University Press 1956.

[6] W.G. Dwyer and H.W. Henn, *Homotopy theoretic methods in group cohomology*, Advanced Courses in Mathematics–CRM Barcelona, Birkhauser Verlag, Basel, 2001.

[7] S. Eilenberg, H. Nagao and T. Nakayama, *On the dimension of modules and algebras IV: dimensions of residue rings of hereditary rings*, Nagoya Math. J. **10** (1956), 87–95.

[8] A.I. Generalov, *Relative homological algebra, cohomology of categories, posets and coalgebras*, pp. 611–638 in M. Hazewinkel (ed.), Handbook of Algebra, Vol. I, Elsevier 1996.

[9] K.W. Gruenberg, *Cohomological topics in group theory*, Lecture Notes in Mathematics **143**, Springer-Verlag (1970).

[10] K.W. Gruenberg, *Relation modules of finite groups*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics **25**, Amer. Math. Soc, Providence RI 1976.

[11] A. Haefliger, *Extension of complexes of groups*, Ann. Inst. Fourier (Grenoble) **42** (1992), 275–311.

[12] P.J. Hilton and U. Stammbach, *A course in homological algebra*, Graduate Texts in Math. **4**, Springer-Verlag 1970.

[13] G. Hoff, *Cohomologies et extensions de categories*, Math. Scand. **74** (1994), 191–207.

[14] M. Linckelmann, *Fusion category algebras*, J. Algebra **277** (2004), 222–235.

[15] M. Linckelmann, *Alperin's weight conjecture in terms of equivariant Bredon cohomology*, Math. Z. **250** (2005), 495–513.

[16] M. Linckelmann, *A note on the Schur multiplier of a fusion system*, J. Algebra **296** (2006), 402–408.

[17] S. MacLane, *Categories for the working mathematician*, Grad. Texts in Math. **5**, Springer-Verlag, New York, 1971.

[18] P.J. Webb, *Standard stratifications of EI categories and Alperin's weight conjecture*, J. Algebra **320** (2005), 4073–4091.

[19] P.J. Webb, *An introduction to the representations and cohomology of categories*, pp. 149-173 in: M. Geck, D. Testerman and J. Thévenaz (eds.), Group Representation Theory, EPFL Press (Lausanne), 2007.

[20] F. Xu, *On the cohomology rings of small categories*, J. Pure Appl. Algebra **212** (2008), 2555–2569.

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