Sets with a Category Action

Peter Webb

1. $\mathcal{C}$-sets

Let $\mathcal{C}$ be a small category and Set the category of sets. We define a $\mathcal{C}$-set to be a functor $\Omega : \mathcal{C} \to \text{Set}$. Thus Ω is simply a diagram of sets, the diagram having the same shape as $\mathcal{C}$: for each object x of $\mathcal{C}$ there is specified a set $\Omega(x)$ and for each morphism $\alpha : x \to y$ there is a mapping of sets $\Omega(\alpha) : \Omega(x) \to \Omega(y)$. If $\mathcal{C}$ happens to be a group (a category with one object and morphism set G) then a $\mathcal{C}$-set is the same thing as a G-set, since the $\mathcal{C}$-set singles out a set and sends each morphism of $\mathcal{C}$ to a permutation of the set. We see that $\mathcal{C}$-sets form a category, the morphisms being natural transformations between the functors. Thus we have a notion of isomorphism of $\mathcal{C}$-sets.

Given two $\mathcal{C}$-sets Ω_1 and Ω_2 we define their *disjoint union* $\Omega_1 \sqcup \Omega_2$ to be the $\mathcal{C}$-set defined at each object x of $\mathcal{C}$ by $(\Omega_1 \sqcup \Omega_2)(x) := \Omega_1(x) \sqcup \Omega_2(x)$ with the expected definition of $\Omega_1 \sqcup \Omega_2$ on morphisms. Let us call a $\mathcal{C}$-set Ω a *single orbit $\mathcal{C}$-set* or *transitive* if it cannot be expressed properly as a disjoint union. A $\mathcal{C}$-set Ω may happen to be the disjoint union of two $\mathcal{C}$-sets, or not; if it can be broken up as a disjoint union we can ask if either of the factors is a disjoint union, and by repeating this we end up with a disjoint union of $\mathcal{C}$-sets each of which is transitive.

Proposition 1.1. *Every finite $\mathcal{C}$-set Ω has a unique decomposition*

$$\Omega = \Omega_1 \sqcup \Omega_2 \sqcup \cdots \sqcup \Omega_n$$

where each Ω_i is transitive. In the diagram

$$\Omega(\mathcal{C}) \xrightarrow{p} \varinjlim \Omega = \{1, \ldots, n\}$$

we may take $\Omega_i(\mathcal{C}) = p^{-1}(i)$.

We present an example. Let $\mathcal{C}$ be the category

$$\mathcal{C} = \underset{x}{\bullet} \xrightarrow{\alpha} \underset{y}{\bullet}$$

which has two objects x and y, a single morphism α from x to y, and the identity morphisms at x and y. We readily see that the transitive (non-empty) $\mathcal{C}$-sets have the form

$$\Omega_n := \underline{n} \to \underline{1}, \quad n \geq 0$$

where $\underline{n} = \{1, \ldots, n\}$ is a set with n elements, the mapping between the two sets sending every element onto a single element. We see various things from this example, such as that a finite category may have infinitely non-isomorphic transitive sets, and also that transitive sets need not be generated by any single element.

We have available another operation on $\mathcal{C}$-sets, namely $\times$. Given two $\mathcal{C}$-sets Ω and Ψ we define $(\Omega \times \Psi)(x) = \Omega(x) \times \Psi(x)$, with the expected definition on morphisms of $\mathcal{C}$. In the above example we see that $\Omega_m \times \Omega_n \cong \Omega_{mn}$.

We are now ready to define the *Burnside ring* of the category $\mathcal{C}$ as

$$B(\mathcal{C}) := \text{Grothendieck group of finite } \mathcal{C}\text{-sets with respect to } \sqcup.$$

Thus $B(\mathcal{C})$ is the free abelian group with the (isomorphism classes of) transitive $\mathcal{C}$-sets as a basis. The multiplication on $B(\mathcal{C})$ is given by $\times$ on the basis elements. Note that this definition of the Burnside ring of a category appears to be quite different to the definitions given by Yoshida in [10] and May in [5].

As an example take $\mathcal{C}$ to be the category which we have seen before. From our calculations we have

$$\begin{aligned} B(\mathcal{C}) &= \mathbb{Z}\{\Omega_0, \Omega_1, \Omega_2, \ldots\} \\ &= \mathbb{Z}\mathbb{N}^{\times}_{\geq 0} \\ &\cong \mathbb{Z}\Omega_0 \oplus \mathbb{Z}\{\Omega_1 - \Omega_0, \Omega_2 - \Omega_0, \ldots\} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}\mathbb{N}^{\times}_{>0} \end{aligned}$$

as rings, where $\mathbb{Z}\mathbb{N}^{\times}_{>0}$ (for example) denotes the monoid algebra over $\mathbb{Z}$ of the multiplicative monoid of non-zero natural numbers. This is the complete decomposition of $B(\mathcal{C})$ as a direct sum of rings. The ring $\mathbb{Z}\mathbb{N}^{\times}_{>0}$ is not finitely generated, and hence neither is $B(\mathcal{C})$.

We illustrate the kind of situation where these constructions may be applied. Quite regularly we consider diagrams of one thing or another, be it sets, or perhaps spaces. By a *space* we mean a simplicial set, in which case a diagram of spaces $\Omega : \mathcal{C} \to \text{Spaces}$ is the same thing as a simplicial $\mathcal{C}$-set. Given such Ω, in each dimension i the i-simplices Ω_i form a $\mathcal{C}$-set. We may form a Lefschetz invariant $\sum_{i\geq 0}(-1)^i \Omega_i$ and this is an element of the Burnside ring $B(\mathcal{C})$. It depends only on the $\mathcal{C}$-homotopy type of Ω. As an example of how this might arise, let G be a finite group and take $\mathcal{C}$ to be the orbit category of G with stablizers in some family of subgroups. Thus the objects of $\mathcal{C}$ are G-sets G/H with H in the specified family, and the morphisms are the equivariant maps. Given a G-space Δ we may obtain a $\mathcal{C}^{\text{op}}$-space $\hat{\Delta}$ by $\hat{\Delta}(G/H) = \Delta^H$, the fixed points, and hence we get a Lefschetz invariant $L(\hat{\Delta})$ in the Burnside ring of the opposite of the orbit category. This invariant carries more information than a similar invariant in the Burnside ring of G considered in [6], [7, p. 358] and [4, Def. 1.6], since the latter invariant is the evaluation of $L(\hat{\Delta})$ at $G/1$.

2. Bisets for categories

Given categories $\mathcal{C}$ and $\mathcal{D}$ we define a $(\mathcal{C}, \mathcal{D})$-*biset* to be the same thing as a $\mathcal{C} \times \mathcal{D}^{\text{op}}$-set. Such a biset Ω is a functor $\mathcal{C} \times \mathcal{D}^{\text{op}} \to \text{Set}$, so given objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$ and morphisms $\alpha : x \to x_1$ in $\mathcal{C}$ and $\beta : y_1 \to y$ in $\mathcal{D}$, and an element $u \in \Omega(x, y)$ we get elements $\alpha u := \Omega(\alpha \times 1_y)(u) \in \Omega(x_1, y)$ and $u\beta := \Omega(1_x \times \beta)(u) \in \Omega(x, y_1)$. In this sense we have an action of $\mathcal{C}$ from the left and $\mathcal{D}$ from the right on Ω.

As an example, we define ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$ to be the $(\mathcal{C}, \mathcal{C})$-biset with ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}(x, y) = \text{Hom}_{\mathcal{C}}(y, x)$, where we reverse the order of x and y because morphisms are composed on the left. In the case of a group this is the regular representation with the group acting by multiplication from the left and from the right.

Given a $(\mathcal{C},\mathcal{D})$-biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ and a $(\mathcal{D},\mathcal{E})$-biset ${}_{\mathcal{D}}\Psi_{\mathcal{E}}$ we construct a $(\mathcal{C},\mathcal{E})$-biset $\Omega\circ\Psi$ by the formula

$$\Omega\circ\Psi(x,z)=\bigsqcup_{y\in\mathcal{D}}\Omega(x,y)\times\Psi(y,z)/\sim$$

where $\sim$ is the equivalence relation generated by $(u\beta,v)\sim(u,\beta v)$ whenever $u\in\Omega(x,y_1)$, $v\in\Psi(y_2,z)$ and $\beta:y_2\to y_1$ in $\mathcal{D}$.

Proving the following result is a very good test of one's understanding of this construction:

Proposition 2.1. *The operation $\circ$ is an associative product, with identity the biset ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$.*

We now define $A(\mathcal{C},\mathcal{D})$ to be the Grothendieck group of finite $(\mathcal{C},\mathcal{D})$-bisets with respect to $\sqcup$, thus extending the notion of the *double Burnside ring* for groups. If R is a commutative ring with 1 we put $A_R(\mathcal{C},\mathcal{D}):=R\otimes_{\mathbb{Z}}A(\mathcal{C},\mathcal{D})$ Using this construction we now define an analog $\mathbb{B}_{\text{Cat}}$ of the *Burnside category* of [1] (see also [2] and [8], for example). The category $\mathbb{B}_{\text{Cat}}$ has as objects all (finite) categories, with homomorphisms given by $\text{Hom}_{\mathbb{B}_{\text{Cat}}}(\mathcal{C},\mathcal{D})=A_R(\mathcal{D},\mathcal{C})$. We define a *biset functor* over R to be an R-linear functor $\mathbb{B}_{\text{Cat}}\to R$-mod. This notion evidently extends the usual notion of biset functors defined on groups, which are R-linear functors defined on the full subcategory $\mathbb{B}_{\text{Group}}$ of $\mathbb{B}_{\text{Cat}}$ whose objects are finite groups.

The Burnside ring functor $B_R(\mathcal{C}):=R\otimes_{\mathbb{Z}}B(\mathcal{C})$ is in fact an example of a biset functor defined on categories. Let $\mathbf{1}$ denote the category with one object and one morphism – in other words, the identity group. We see that if $\mathcal{C}$ is any category, $\mathcal{C}$-sets may be identified as the same thing as $(\mathcal{C},\mathbf{1})$-bisets, so that $B_R(\mathcal{C})=A_R(\mathcal{C},\mathbf{1})=\text{Hom}_{\mathbb{B}_{\text{Cat}}}(\mathbf{1},\mathcal{C})$. Thus B_R is a representable biset functor over R, and hence it is projective. It is indecomposable since its endomorphism ring is $\text{End}(B_R)\cong A_R(\mathbf{1},\mathbf{1})\cong R$ by Yoneda's lemma (assuming R is indecomposable).

All this is similar to what happens with biset functors defined on groups, as described in [2], and the story continues. Supposing that the ring R we work over is a field or complete discrete valuation ring, for formal reasons the simple biset functors may be parametrized by pairs $(\mathcal{C},V)$ consisting of a category $\mathcal{C}$ and a simple $\text{End}_{\mathbb{B}_{\text{Cat}}}(\mathcal{C})$-module V, subject to a certain equivalence relation described in a slightly different context in [9, Cor. 4.2]. Each simple functor $S^{\text{Cat}}_{\mathcal{C},V}$ has a projective cover $P^{\text{Cat}}_{\mathcal{C},V}$: an indecomposable projective with $S^{\text{Cat}}_{\mathcal{C},V}$ as its unique simple quotient. Because the category of groups is a full subcategory of the category of small categories the relationship between functors defined on $\mathbb{B}_{\text{Cat}}$ and $\mathbb{B}_{\text{Group}}$ is similar to that of representations of an algebra Λ and of $e\Lambda e$ where $e\in\Lambda$ is idempotent. This kind of relationship was described by Green in [3] is described in a context close to the present one in sections 3 and 4 of [9]. Some of this relationship goes as follows.

Proposition 2.2. *Let S be a simple biset functor defined on categories. Then its restriction to groups is either zero or a simple functor and establishes a bijection $S_{G,V}^{\mathrm{Cat}} \leftrightarrow S_{G,V}^{\mathrm{Group}}$ between isomorphism types of simple biset functors defined on categories which are non-zero on groups, and simple biset functors defined on groups G. Furthermore $P_{G,V}^{\mathrm{Cat}} \downarrow_{\mathrm{Group}}^{\mathrm{Cat}} \cong P_{G,V}^{\mathrm{Group}}$, and $P_{G,V}^{\mathrm{Group}} \uparrow_{\mathrm{Group}}^{\mathrm{Cat}} \cong P_{G,V}^{\mathrm{Cat}}$ where $\uparrow_{\mathrm{Group}}^{\mathrm{Cat}}$ denotes the left adjoint to the restriction $\downarrow_{\mathrm{Group}}^{\mathrm{Cat}}$.*

Thus every simple biset functor defined on groups extends uniquely to a simple biset functor defined on categories, and the same holds for indecomposable projective biset functors. We see, when R is a field, that the Burnside ring functor B_R is in fact the indecomposable projective $P_{\mathbf{1},R}^{\mathrm{Cat}}$ with unique simple quotient $S_{\mathbf{1},R}^{\mathrm{Cat}}$.

We conclude by mentioning that the values of this simple functor may be identified in terms of a certain bilinear pairing between the Burnside ring of a category and of its opposite, generalizing a bilinear form introduced in [2]. The Burnside ring $B_R(\mathcal{C})$ has as basis the transitive $(\mathcal{C},\mathbf{1})$-bisets ${}_{\mathcal{C}}\Omega_{\mathbf{1}}$, and $B_R(\mathcal{C}^{\mathrm{op}})$ has as basis the transitive $(\mathbf{1},\mathcal{C})$-bisets ${}_{\mathbf{1}}\Psi_{\mathcal{C}}$. We define a bilinear map $\langle\ \ ,\ \ \rangle : B_R(\mathcal{C}^{\mathrm{op}}) \times B_R(\mathcal{C}) \to R$ by $\langle {}_{\mathbf{1}}\Psi_{\mathcal{C}}, {}_{\mathcal{C}}\Omega_{\mathbf{1}} \rangle = |{}_{\mathbf{1}}\Psi_{\mathcal{C}} \circ {}_{\mathcal{C}}\Omega_{\mathbf{1}}|$, the size of this set.

Proposition 2.3. *If R is a field then the dimension of the simple biset functor $S_{\mathbf{1},R}$ is the rank of the above bilinear pairing.*

The observations here are just the start of a development of theory on which the author is currently working.

References

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