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A local method in group cohomology

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1. Introduction

Let G be a finite group. The classical approach to the local control of the cohomology of G is described in the book of Cartan and Eilenberg [5] and relies on the fact that for any prime p the Sylow p-subgroup of $H^n(G, M)$ is isomorphic to the subgroup of "stable elements" of $H^n(P, M)$ under the action of G, where P is a Sylow p-subgroup of G. In some situations the computation of the stable elements has been reduced to a local problem, as in for example [14] and [16]. When this is done it is usual to make restrictions that the action of G on M is trivial, and either that G has a special structure or that the prime p is sufficiently large. However, in the most general situation it seems hard to make this method work, the problem being to compute the conjugation action of G on $H^n(P, M)$. We present a completely different approach to the local calculation of cohomology which avoids these restrictions. Our method has some connections with equivariant cohomology in that we consider a group acting on a simplicial complex and we obtain the cohomology of G in terms of the cohomology of the isotropy groups. But our approach is mostly algebraic, and we seem to obtain sharper results than are usually obtained with equivariant cohomology.

Our first theorem concerns the abstract situation of a finite group G acting on a simplicial complex Δ, and we later go on to give the applications to particular cases. We assume G acts simplicially, and denote the isotropy group $\{g \in G \mid \sigma g = \sigma\}$ by G_σ. Assume further that G_σ fixes the vertices of σ pointwise. This can always be achieved by passing to a barycentric subdivision if necessary. Throughout this paper we will let $\mathscr{C}$ denote the collection of subgroups H of G which have a normal p-subgroup with a cyclic p' quotient, that is

$$\mathscr{C} = \{H \leqslant G \mid H/O_p(H) \text{ is cyclic}\}.$$

The subgroups in $\mathscr{C}$ are sometimes called "cyclic mod p."

THEOREM A. *Let G act simplicially on the simplicial complex Δ, suppose for each simplex σ the isotropy group G_σ fixes σ pointwise, and let p be a fixed* *prime. Assume that one of the following conditions holds:*

(*a*) *for each* $H \in \mathscr{C}$ *with* $O_p(H) \neq 1$ *the fixed point complex* Δ^H *has Euler characteristic* $\chi(\Delta^H) = 1$

or (*b*) *for each cyclic subgroup* H *of order* p, Δ^H *is acyclic. Then for any* $\mathbb{Z}G$*-module* M *and integer* n,

$$\hat{H}^n(G, M)_p = \sum_{\sigma \in \Delta/G} (-1)^{\dim(\sigma)} \hat{H}^n(G_\sigma, M)_p.$$

In the statement of Theorem A we use the suffix p to indicate the Sylow p-subgroup of the corresponding cohomology group. The sum on the right is taken over a set of representatives for the orbits of G on Δ and the alternating sum is to be understood in the Grothendieck group of finite abelian groups with relations given by direct sum decompositions. It can also be interpreted by transferring the groups with negative sign over to the left hand side with a corresponding positive coefficient. There is then an isomorphism between the direct sums of the groups on either side. It is plain that this is sufficient to determine the isomorphism type of $\hat{H}^n(G, M)_p$ provided M is a finitely generated $\mathbb{Z}G$-module, since then all the cohomology groups are finite. In fact condition (*b*) implies a condition (*a*), as we shall see, but it is usually easier to verify, for example by showing that Δ^H is contractible if $|H| = p$. The conclusion of Theorem A does not hold in this generality for the ordinary H^0 and H_0; instead we must take Tate cohomology, which of course includes the usual homology groups in dimensions ≤ -2.

In the applications of Theorem A the simplicial complex Δ always arises from a partially ordered set in the following standard fashion: if S is a poset, the associated simplicial complex has as its n-simplices the chains $s_0 < \cdots < s_n$ of length $n+1$ of elements of S. The faces of such an n-simplex are the subchains of shorter length. If G acts on S there is an induced action on the simplicial complex, and it is clear that the isotropy group G_σ will stabilize all the vertices of σ, since it must fix all elements of S in the chain σ. We will use the symbols $\mathscr{A}$ to denote the poset of all non-identity elementary abelian p-subgroups of G and $\mathscr{S}$ to denote the poset of non-identity p-subgroups of G. We regard these also as simplicial complexes by the above construction, and G acts on all of them by means of conjugating the subgroups. Our results apply to these simplicial complexes and also to the Tits building of a finite Chevalley group. This can be regarded as the simplicial complex obtained from the poset of proper parabolic subgroups of G. For each of these complexes the condition that Δ^H is contractible and hence acyclic when $|H| = p$ has been verified by Quillen. We therefore obtain:

THEOREM B. *The cohomology formula in Theorem A is a valid when $\Delta = \mathscr{A}$, $\mathscr{S}$, or Δ is the Tits building of a finite Chevalley group in defining characteristic p.*

In the case of a Tits building, if we choose the simplices to be the proper parabolic subgroups themselves, rather than chains of subgroups, the formula adopts the following form.

COROLLARY C. *Let G be a finite Chevalley group in defining characteristic p and let B be a fixed Borel subgroup. Then*

$$\hat{H}^n(G, M)_p = \sum_{P \supseteq B} (-1)^{\text{rank}(P)} \hat{H}^n(P, N)_p.$$

The sum here is taken over all parabolic subgroups containing B. By the rank of a parabolic subgroup P we mean that integer m so that $P = P_m < P_{m-1} < \cdots < P_1 < P_0$ is a chain of proper parabolics of maximum length. Maximal parabolic subgroups thus have rank 0.

In many cases evaluation of the formula for cohomology in Theorem A can be quite straightforward. If Δ arises from a poset of subgroups of G as in Theorem B then the simplices of dimension zero are just the subgroups themselves, and the stabilizers G_σ are the normalizers of the subgroups. In general, if σ is a simplex $H_0 < \cdots < H_n$ where the H_i are subgroups of G, then $G_\sigma = N_G(H_1) \cap \cdots \cap N_G(H_n)$. Observe that for each of the posets $\mathscr{A}$ and $\mathscr{S}$, the maximal p-local subgroups always appear amongst the isotropy groups G_σ, because for example with $\mathscr{A}$, a p-local subgroup always normalizes some elementary abelian p-subgroup. We can say in general that the p-part of the cohomology of G is determined by the cohomology of certain p-local subgroups, and their intersections. However, if G has a normal p-subgroup then G itself will appear on the right hand side of the cohomology formula with these three posets, and Theorem B is of less use. In this situation we can apply a different reduction theorem which emerges as a step in the proof of Theorem A.

THEOREM D. *Let $\mathscr{X}$ be a class of subgroups of G which is closed under taking conjugates and forming subgroups, and with $\mathscr{X} \supseteq \mathscr{C}$. Then*

$$H^n(G, M)_p = \sum_{H \in \mathscr{X}} \frac{f(H)}{|G:H|} H^n(H, M)_p$$

for any integer n and $\mathbb{Z}G$-module M, where $f:\mathscr{X}\to\mathbb{Z}$ is the function defined by the equations

$$\sum_{J\le K\in\mathscr{X}} f(K)=1 \quad \textit{for every} \quad J\in\mathscr{X}.$$

This theorem works both for Tate and ordinary cohomology and homology (the only difference between the two being the groups H^0, H_0 and the corresponding Tate groups $\hat{H}^0$ and $\hat{H}^{-1}$).

There is another way to view the function f; if we let $\hat{\mathscr{X}}$ be the poset $\mathscr{X}\cup\{\infty\}$ where ∞ is an artificial maximal element, then $f(H)=-\mu(H,\infty)$, where μ is the Möbius function. It is often an elementary but time-consuming matter to compute the values of f from the defining equations in Theorem D. Evidently if H is a maximal member of $\mathscr{X}$ then $f(H)=1$, and by working down through chains of subgroups from these maximal members we may build up further values of f. Because the Möbius function is zero except on intersections of maximal elements, these intersections are the only subgroups we need consider. At the end of §2 we present a formalized version of the inductive computation procedure just hinted at, and this may be suitable for machine computation.

We prove Theorem A by obtaining an isomorphism between direct sums of certain permutation modules. Let $\mathbb{Z}_p$ denote the p-adic integers, and if H is a subgroup of G, write u_H for the corresponding permutation module $\mathbb{Z}_p\otimes_{\mathbb{Z}_pH}\mathbb{Z}_pG$ over $\mathbb{Z}_pG$.

THEOREM A′. $u_G\equiv\sum_{\sigma\in\Delta/G}(-1)^{\dim(\sigma)}u_{G_\sigma}$ *(modulo projectives).*

Here the alternating sum is taken in the Green ring of finitely generated $\mathbb{Z}_pG$-modules, but it may also be treated in a similar manner to the sum in Theorem A. The congruence modulo projectives means that we may achieve equality by adding a suitable finitely generated projective module to each side.

There is a theorem analogous to Theorem D concerning permutation modules.

THEOREM D′. *With $\mathscr{X}$ and f defined as in Theorem D,*

$$u_G=\sum_{H\in\mathscr{X}} u_H\frac{f(H)}{|G:H|}.$$

The above equation holds in the Green ring defined over $\mathbb{Q}$ of $\mathbb{Z}_pG$-modules.

The formula in Theorem A reminds one of an Euler characteristic, and in particular the theorem of Brown (as improved by Quillen) [3, p. 267] that

$$\chi(G) \equiv \sum_{\sigma \in \mathcal{A}/G} (-1)^{\dim \sigma} \chi(G_\sigma) \pmod{\mathbb{Z}_p}$$

valid for certain groups with $\mathrm{vcd}(G) < \infty$, where $\mathcal{A}$ is the complex of elementary abelian p-subgroups. In §§5 and 6 we investigate the connection between our approach and the method which Brown used, namely equivariant cohomology. It would be satisfying to prove Theorem A using the spectral sequence of equivariant cohomology, but I have been unable to do this in general without assuming some further properties of the simplicial complex on which G acts. Under slightly stronger conditions on Δ than those in Theorem A we prove that Δ has these further properties, provided Δ is a graph. This happens for $\Delta = \mathcal{A}$ when G has p-rank 2, that is, the largest elementary abelian p-subgroup of G is $C_p \times C_p$. The properties we require are summarised in the next result, and they immediately give a proof of Theorem A in this case.

THEOREM E. *Let G be a group of p-rank 2, and let $\Delta = \mathcal{A}$ or $\mathcal{S}$. Then for each r the p-adic completion $\tilde{H}_r(\Delta)_p$ is a projective $\mathbb{Z}_pG$-module, where $\tilde{H}_r$ denotes reduced homology. If $C_1 \xrightarrow{d} C_0$ is the chain complex of $\mathcal{A}$ then both of the short exact sequences*

$$0 \to H_1(\mathcal{A})_p \to (C_1)_p \to \mathrm{Im}\,(d)_p \to 0$$

and

$$0 \to \mathrm{Im}\,(d)_p \to (C_0)_p \to H_0(\mathcal{A})_p \to 0$$

are split as sequences of $\mathbb{Z}_pG$-modules.

In 5.1 as a stage in the proof of Theorem E we state a similar result valid for arbitrary connected graphs Δ under a certain contractibility hypothesis. This result seems to be known, but it is perhaps of some combinatorial interest since group actions on graphs of the required kind do arise in practice, an important example being a finite Chevalley group acting on a Tits building when this is a graph. In this case the fact that the first homology group is projective at the prime p is well-known, because it is the Steinberg module. Theorem 5.1 can be regarded as an extension of this fact for arbitrary groups. A straightforward consequence is that the rank of $H_1(\Delta)$ is divisible by the order of a Sylow p-subgroup of G.

In §7 we calculate the isotropy groups in the formula in Theorem A for various specific cases, and in several of these the information given about the cohomology is new. We conclude in §8 with some formulae which have the nature of an Euler characteristic and involve numbers such as the group order, or convey certain local information about G.

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2. Reduction to local subgroups and the proof of Theorem D

The proofs we shall give of Theorems A and D have rather little to do with cohomology, and rely on establishing Theorems A′ and D′, which are results in representation theory. We first make some remarks about the Grothendieck groups in which the equations in these theorems are supposed to hold. To handle the equations between permutation modules we work in the representation ring of $\mathbb{Z}_pG$-modules. This is the vector space $A(G)$ over the rational numbers with the set of isomorphism classes of finitely generated indecomposable $\mathbb{Z}_pG$-modules as a basis, allowing both torsion and torsion free modules. If $M = M_1 \oplus \cdots \oplus M_n$ is any finitely generated $\mathbb{Z}_pG$-module where the M_i are indecomposable, we associate to M the corresponding element $M_1 + \cdots + M_n$ in $A(G)$. We will fail to distinguish in our notation between a module and its isomorphism class. Because of the Krull-Schmidt theorem (see [26]) the choice of the element $M_1 + \cdots + M_n$ representing M is uniquely determined. There is a product in $A(G)$ defined on basis elements by $M \cdot N = M \otimes_{\mathbb{Z}_p} N$, and the identity element is $\mathbb{Z}_p$. Our aim is to obtain an alternative expression for $\mathbb{Z}_p$, and we do this using Conlon's induction theorem and a formula for idempotents in $A(G)$ which arise from the Burnside algebra. The expressions involving sums of cohomology groups on the right hand sides of the equations in Theorems A and D hold inside the Grothendieck group of finite abelian p-groups with respect to direct sum decompositions, tensored up to $\mathbb{Q}$. This is a subspace of $A(1)$, and it was the observation that $H^n(G,\ )$ preserves finite direct sums and hence induces a homomorphism

$$H^n(G,\ ): A(G) \to A(1)$$

which provided a starting point for this research. A similar observation was made by Roggenkamp and Scott [20].

Continuing with the notation for permutation modules $u_H = \mathbb{Z}_p \otimes_{\mathbb{Z}_pH} \mathbb{Z}_pG$ used in the Introduction, we indicate how Theorems A and D may be deduced from A′ and D′. It depends on the isomorphisms

$$\mathrm{Ext}^n_{\mathbb{Z}_pG}(u_H, M_p) = \mathrm{Ext}^n_{\mathbb{Z}_pG}(\mathbb{Z}_p \otimes_{\mathbb{Z}_pH} \mathbb{Z}_pG, M_p) \cong \mathrm{Ext}^n_{\mathbb{Z}_pH}(\mathbb{Z}_p, M_p)$$
$$\cong \mathrm{Ext}^n_{\mathbb{Z}H}(\mathbb{Z}, M) \otimes_{\mathbb{Z}} \mathbb{Z}_p = H^n(H, M)_p.$$

The penultimate isomorphism where we take the completion at p outside the Ext term is valid because $\mathbb{Z}$ is finitely presented as a $\mathbb{Z}H$-module, and completion at p is an exact functor [12, p. 233]. If we thus apply $\mathrm{Ext}^n_{\mathbb{Z}_pG}(\ , M_p)$ to both sides of Theorem D′ we immediately obtain Theorem D, at least when $n \geq 1$. For the zero cohomology and homology groups the formula follows in the same way using the functors $\mathrm{Hom}_{\mathbb{Z}_pG}(\ , M_p)$ and $\otimes_{\mathbb{Z}_pG} M_p$. We may deduce the result for Tate groups $\hat{H}^n(G, M)$ when $n \leq 0$ by applying dimension shifting, as in the formula $\hat{H}^n(G, M) \cong \hat{H}^{n+s}(G, \Omega^s(M))$, where Ω is the Heller operator (see [26]). Thus for $n \leq 0$ we take $s = -n + 1$ and the desired formula for $\hat{H}^n(G, M)$ is identical with the corresponding formula for $\hat{H}^1(G, \Omega^{-n+1}(M))$. We should note in passing that Ω commutes up to projective summands (on which $\hat{H}^1$ vanishes) with restricting M to H. Theorem A follows similarly from Theorem A′, except that here we start off with a congruence modulo projectives. We obtain an equality in cohomology because the Ext groups which have a projective module in the first place are all zero. The dimension shifting argument works again to get the negative Tate groups, but note that Theorem A does not hold in general for H^0 and H_0.

In the remainder of this section we give a proof of Theorem D′ and discuss its uses. We introduce the Burnside algebra, $B(G)$, which Burnside considered in his book. This is the $\mathbb{Q}$-vector space with the set of equivalence classes of transitive G-sets as a basis. A transitive G-set is a set of cosets $H\backslash G$, and $H\backslash G$ is equivalent to $K\backslash G$ if and only if H and K are conjugate subgroups of G. The product on $B(G)$ is given on the basis elements by taking the direct product of the corresponding G-sets and expressing it as a disjoint union of transitive G-sets. This determines a linear combination of the basis elements according to the multiplicities with which they occur in the disjoint union, and this is defined to be the product. We will use the same symbol u_H for the G-set $H\backslash G$ that we previously used for the corresponding permutation module $\mathbb{Z}_p(H\backslash G) \cong \mathbb{Z}_p \otimes_{\mathbb{Z}_pH} \mathbb{Z}_pG$, even though it may happen that $u_H = u_K$ in $A(G)$ but $u_H \neq u_K$ in $B(G)$ for different subgroups H and K [Scott, unpublished]. There is, however, a homomorphism $B(G) \to A(G)$ sending u_H as $H\backslash G$ to u_H as the permutation module $\mathbb{Z}_p(H\backslash G)$. $B(G)$ is a semisimple algebra [22], and for each subgroup $H \leq G$ there is an idempotent e_H lying in the span of $\{u_k \mid K \leq H\}$. These idempotents are given by the following formula:

2.1. THEOREM (Gluck [11], Yoshida [25]). *In* $B(G)$,

$$e_H = \frac{1}{|N_G(H)|} \sum_{K \leqslant H} \mu(K, H) \cdot |K| \cdot u_K$$

where μ *is the Möbius function on the lattice of subgroups of* G *with defining property*

$$\sum_{J \leqslant K \leqslant H} \mu(K, H) = \delta_{JH} \quad \textit{for all subgroups } H, J \textit{ of } G$$

(*Kronecker delta*).

The e_H form a complete set of primitive idempotents in $B(G)$, and we have

$$u_G = 1 = \sum_H e_H$$

the sum taken over all conjugacy classes of subgroups H. Applying the homomorphism from $B(G)$ to $A(G)$ we obtain an identical formula there. In $A(G)$ we may perform some simplification, because many of the e_H are zero.

As in §1 we define

$$\mathscr{C} = \{H \leqslant G \mid H/O_p(H) \text{ is cyclic}\}$$

and call the subgroups in $\mathscr{C}$, "cyclic mod p", If $\mathscr{X}$ is any class of subgroups of G closed under conjugation, let $\mathscr{X}^*$ denote a set of representatives for the conjugacy classes. Note that $\mathscr{C}$ is itself closed under taking subgroups (and under conjugation).

2.2. THEOREM (Conlon [8]). *Under the canonical homomorphism* $B(G) \to A(G)$ *the idempotent* e_H *is mapped to zero if and only if* $H \notin \mathscr{C}$.

This allows us to throw away terms in the expression $u_G = \sum_H e_H$ and we obtain

2.3. COROLLARY. *Suppose* $\mathscr{X}$ *is a class of subgroups of* G *closed under* *conjugation and with $\mathscr{X} \supseteq \mathscr{C}$. Then in $A(G)$,*

$$u_G = \sum_{H \in \mathscr{X}^*} e_H.$$

We may evidently substitute the formula for e_H given in 2.1 into the expression in 2.3 to obtain our identity between the u_H in $A(G)$. By doing this and rearranging the double sum we obtain Theorem D′.

Proof of Theorem D′

$$\begin{aligned} u_G = \sum_{H \in \mathscr{X}^*} e_H &= \sum_{H \in \mathscr{X}^*} \frac{1}{|N_G(H)|} \sum_{K \leq H} \mu(K, H) \cdot |K| \cdot u_K \\ &= \sum_{K \leq H \in \mathscr{X}} \frac{1}{|G : N_G(H)|} \cdot \frac{1}{|N_G(H)|} \mu(K, H)\, |K|\, u_K \\ &= \frac{1}{|G|} \sum_{K \leq H \in \mathscr{X}} \mu(K, H)\, |K|\, u_K = \frac{1}{|G|} \sum_{K \in \mathscr{X}} u_K \cdot |K| \cdot \Big(\sum_{K \leq H \in \mathscr{X}} \mu(K, H) \Big). \end{aligned}$$

We now define $f(K) = \sum_{K \leq H \in \mathscr{X}} \mu(K, H)$. Then for any fixed $J \in \mathscr{X}$,

$$\sum_{J \leq K \in \mathscr{X}} f(K) = \sum_{J \leq K \leq H \in \mathscr{X}} \mu(K, H) = \sum_{J \leq H \in \mathscr{X}} \delta_{JH} = 1,$$

and it is apparent that these equations suffice to determine the values of f completely. This proves Theorem D′.

Remark. If H and K are conjugate subgroups of G then $u_H \cong u_K$. Thus u_H appears $|G : N_G(H)|$ times in the sum in Theorem D′, and hence we may rewrite it as

$$u_G = \sum_{H \in \mathscr{X}^*} u_H \frac{f(H)}{|N_G(H) : H|}$$

A similar modification to Theorem D is possible.

The computation of values of the function f is a rather mechanical process and can profitably be done with computer assistance. The author has found the following scheme to be quite economical. For each pair of subgroups J, K in $\mathscr{X}$, define C_{JK} to be the number of conjugates of K which contain J. This number does not depend on the choice of J or K within their conjugacy classes. Then for a fixed $J \in \mathscr{X}$ the defining equation

$$\sum_{J \leq K \in \mathscr{X}} f(K) = 1$$

becomes

$$\sum_{J \leq K \in \mathscr{X}^*} f(K) C_{JK} = 1.$$

If we define the matrix $C = (C_{JK})_{J,K \in \mathscr{X}^*}$ and the column vector $f = (f(K))_{K \in \mathscr{X}^*}$, the last equation is

$$C \cdot f = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

The problem is now to compute the matrix C, and this is usually best done by noting that

$$C_{JK} = \frac{|G : N_G(K)|}{|G : N_G(J)|} \cdot C'_{JK}$$

where C'_{JK} is the number of conjugates of J contained in K. This latter equation may be verified by considering the bipartite graph whose vertices are the conjugates of J and the conjugates of K, and J^x is joined to K^y if and only if $J^x \subseteq K^y$. The number of edges in the graph may be computed in two ways as $C_{JK}\,|G : N_G(J)|$ and $C'_{JK}\,|G : N_G(K)|$. Finally, the solution of the matrix equation is elementary, since by placing the elements of $\mathscr{X}^*$ in non-decreasing order, C is a triangular matrix.

We give an example of the above calculation when $G = \Sigma_4$, $p = 2$ and $\mathscr{X} = \mathscr{C}$. Representatives of the conjugacy classes of $\mathscr{C}$ are $A_4, D_8, V = \langle (12)(34), (13)(24) \rangle$, $\langle (12), (34) \rangle$, C_4, C_3, $\langle (12) \rangle$, $\langle (12)(34) \rangle$ and with rows and columns corresponding to these subgroups in the given order we have

$$C = \begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 1 & 3 & 1 &&&&& \\ 0 & 1 & 0 & 1 &&&& \\ 0 & 1 & 0 & 0 & 1 &&& \\ 1 & 0 & 0 & 0 & 0 & 1 && \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & \\ 1 & 3 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The upper triangle is zero. Solving the equation $C \cdot f = (1, \ldots, 1)^t$ gives $f = (1, 1, -3, 0, 0, 0, 0, 0)^t$, so that the vector with entries $f(K)/\,|N_G(K):K|$ is $(\frac{1}{2}, 1, -\frac{1}{2}, 0, 0, 0, 0, 0)^t$. The statement of Theorem B′ is

$$u_{\Sigma_4} = \tfrac{1}{2}u_{A_4} + u_{D_8} - \tfrac{1}{2}u_V.$$

Passing to cohomology we obtain

$$H^n(\Sigma_4, M)_2 = \tfrac{1}{2}H^n(A_4, M)_2 + H^n(D_8, M)_2 - \tfrac{1}{2}H^n(V, M)_2.$$

Some simplifications of the above procedure are possible. If we are only interested in cohomology it is a waste of time to compute the u_H where $p \nmid |H|$, since these are projective $\mathbb{Z}_pG$-modules and have trivial cohomology. Thus we might as well omit such subgroups from the matrix C. Secondly, one sees that $f(K) = 0$ except when K is expressible as an intersection of maximal members of $\mathscr{X}$. The shortest way to establish this is to interpret f as a Möbius function, as was indicated in §1. We let $\hat{\mathscr{X}}$ be the poset $\mathscr{X} \cup \{\infty\}$, where ∞ is greater than every member of $\mathscr{X}$. The Möbius function on $\hat{\mathscr{X}}$ is then defined by

$$\sum_{J \le K \le \infty} \mu(K, \infty) = 0 \quad \text{for every} \quad J \ne \infty$$

$$\mu(\infty, \infty) = 1$$

and evidently these equations are satisfied if we take $\mu(K, \infty) = -f(K)$. This Möbius function should not be confused with the Möbius function on the lattice of subgroups of G. It was proved by Philip Hall [13] that $\mu(K, \infty) = 0$ unless K is an intersection of maximal elements, hence our assertion. With this observation we could have said immediately that A_4, D_8 and V are the only subgroups which make a non-zero contribution in the calculation for Σ_4.

We wish to conclude this section by describing another way in which formulae for u_G can be obtained. If G is not itself cyclic mod p (i.e. $G/O_p(G)$ is not cyclic) then $e_G = 0$ in $A(G)$ by Conlon's Theorem 2.2. In Gluck's and Yoshida's expression for e_G (Theorem 2.1) the coefficient of u_G is $\mu(G, G) = 1$, so we may write

$$u_G = -\frac{1}{|G|} \sum_{K < G} |K|\, \mu(K, G) \cdot u_K.$$

We obtain

2.4. PROPOSITION. *$H^n(G, M)_p = -1/|G| \sum_{K<G} |K| \mu(K, G) H^n(K, M)_p$ for every prime p such that $G/O_p(G)$ is non-cyclic. Observe that we also obtain this result from Theorem D on taking $\mathscr{X}$ = all proper subgroups of G.*

In [13], Hall used the notation $\mu(K)$ for our Möbius function $\mu(K, G)$, and he computed values of $\mu(K)$ for some particular groups. Thus, for example, he gives the following Möbius inversion formula for $GL(3, 2) = G_{168}$:

$$\begin{aligned}\phi(G_{168}) = \sigma(G_{168}) - 7\sigma(O_{24}) - 7\sigma(O_{24}) - 8\sigma(M_{7,3}) + 21\sigma(O_8)\\ + 28\sigma(D_6) + 56\sigma(C_3) - 84\sigma(C_2).\end{aligned}$$

Hall's notation for the subgroups of $GL(3, 2)$ is O_{24} for the octahedral group of order 24, $M_{7,3}$ for the non-abelian group of order 21, O_8 for dihedral of order 8, D_6 for dihedral of order 6, and C_3, C_2 for cyclic groups. We have written $-14\sigma(O_{24})$ as $-7\sigma(O_{24}) - 7\sigma(O_{24})$ because there are two conjugacy classes of these subgroups. Hall did not need to distinguish between the conjugacy classes in this way. To explain the rest of his notation it is sufficient for our purposes to say that there are (for example) 8 maximal subgroups of type $M_{7,3}$, for each of which $\mu(M_{7,3}) = -1$, so we obtain a term $-8\sigma(M_{7,3})$ in the above expression. Here ϕ and σ are functions defined on the subgroups of G_{168} satisfying $\sigma(H) = \sum_{K \leqslant H} \phi(K)$. We immediately read off the following formula in cohomology:

$$\begin{aligned}H^n(G, M)_2 &= \tfrac{1}{168}[7.24H^n(O_{24}, M)_2 + 7.24H^n(O_{24}, M)_2 - 21.8H^n(O_8, M)_2\\ &\quad - 28.6H^n(D_8, M)_2 + 84.2H^n(C_2, M)_2]\\ &= H^n(O_{24}, M)_2 + H^n(O_{24}, M)_2 - H^n(O_8, M)_2 - H^n(D_6, M)_2\\ &\quad + H^n(C_2, M)_2.\end{aligned}$$

Since we have decided to compute the 2-parts of cohomology groups we have omitted the terms with subgroups of odd order. It is well known, and easy to prove, that $H^n(D_6, M)_2 = H^n(C_2, M)_2$, so the last two terms above cancel, and we obtain

$$\begin{aligned}H^n(G, M)_2 &= H^n(O_{24}, M)_2 + H^n(O_{24}, M)_2 - H^n(O_8, M)_2\\ &= \tfrac{1}{2}H^n(A_4, M)_2 + \tfrac{1}{2}H^n(A_4, M)_2 + H^n(O_8, M)_2 - \tfrac{1}{2}H^n(V, M)_2\\ &\quad - \tfrac{1}{2}H^n(V, M)_2.\end{aligned}$$

The second line is obtained by inserting the formula previously obtained for O_{24}.

Here $V \cong C_2 \times C_2$, and repeated terms with the same subgroup indicate different conjugacy classes. Using a formula such as this one may readily compute the Poincaré series of the cohomology ring $\oplus_{n=0}^{\infty} H^n(G, F_2)$.

3. Proof of Theorems A and A′

As explained in §2, Theorem A follows from Theorem A′ by applying $\mathrm{Ext}^n_{\mathbb{Z}_pG}(\ , M_p)$ to both sides of the congruence of Theorem A′. Since Ext is zero on projective modules, the congruence becomes an equality between cohomology groups. We therefore prove Theorem A′.

The idea behind the proof of Theorem A′ is as follows. For each term u_{G_σ} which appears in the congruence we have to verify, we obtain by Theorem D′ an expression in terms of the u_H where H is cyclic mod p and is a subgroup of G_σ. We will substitute these expressions into both sides of the congruence in Theorem A′ and after some rearrangement of the terms we will show that the two sides are equal. With this end in view we use the notation $\mathscr{C}(G_\sigma)$ for those subgroups of G_σ which are cyclic mod p, so that $\mathscr{C}(G) = \mathscr{C}$. Evidently $\mathscr{C}(G_\sigma) = \mathscr{C} \cap \{\text{all subgroups of } G_\sigma\}$. For each subgroup G_σ there will be a function f defined on $\mathscr{C}(G_\sigma)$. We now denote this function by f_σ, retaining the symbol f for the function on $\mathscr{C}$. For each subgroup G_σ Theorem D′ gives an identity

$$\mathbb{Z}_p = \sum_{H \in \mathscr{C}(G_\sigma)} \frac{f_\sigma(H)}{|G_\sigma : H|} \cdot \mathbb{Z}_p \uparrow_H^{G_\sigma}$$

in $A(G_\sigma)$. We use the up arrow to denote induction. Inducing this up to G we obtain

$$u_{G_\sigma} = \sum_{H \in \mathscr{C}(G_\sigma)} \frac{f_\sigma(H)}{|G_\sigma : H|} u_H$$

in $A(G)$. The right hand side of the equation in Theorem A′ is

$$\begin{aligned}
\sum_{\sigma \in \Delta/G} (-1)^{\dim(\sigma)} u_{G_\sigma} &= \sum_{\sigma \in \Delta} \frac{(-1)^{\dim(\sigma)}}{|G : G_\sigma|} u_{G_\sigma} \\
&= \sum_{\sigma \in \Delta} \frac{(-1)^{\dim(\sigma)}}{|G : G_\sigma|} \sum_{H \in \mathscr{C}(G_\sigma)} \frac{f_\sigma(H)}{|G_\sigma : H|} u_H \\
&= \sum_{\substack{H \in \mathscr{C}(G_\sigma) \\ \sigma \in \Delta}} \frac{(-1)^{\dim(\sigma)} f_\sigma(H)}{|G : H|} u_H \\
&= \sum_{H \in \mathscr{C}(G)} \frac{u_H}{|G : H|} \Bigg(\sum_{\substack{G_\sigma \supseteq H \\ \sigma \in \Delta}} (-1)^{\dim(\sigma)} f_\sigma(H) \Bigg).
\end{aligned}$$

By Theorem D′, the left hand side of the equation in Theorem A′ is

$$u_H = \sum_{H \in \mathscr{C}(G)} \frac{u_H}{|G:H|} f(H).$$

We will show that provided $O_p(H) \neq 1$ and $H \in \mathscr{C}(G)$,

$$f(H) = \sum_{\substack{G_\sigma \supseteq H \\ \sigma \in \Delta}} (-1)^{\dim(\sigma)} f_\sigma(H).$$

This will suffice to prove Theorem A′, since the two sides of the congruence in the theorem differ by at most a linear combination of the u_H, $H \in \mathscr{C}(G)$, where $O_p(H) = 1$. Such a subgroup H has order prime to p, so u_H is projective.

To verify the last equation above we check that the right hand side satisfies the defining property of f, namely

$$\sum_{H \leqslant K \in \mathscr{C}(G)} f(K) = 1 \quad \text{for all} \quad H \in \mathscr{C}(G)$$

provided that $O_p(H) \neq 1$. We wish to use this to define $f(H)$ inductively when $O_p(H) \neq 1$ and when $f(K)$ has already been defined if $H < K \in \mathscr{C}(G)$. This is valid, since in this situation $O_p(K) \neq 1$. Note that the condition $G_\sigma \supseteq H$ in the equation to be verified is equivalent to saying σ is fixed by H, i.e. $\sigma \in \Delta^H$. We calculate

$$\sum_{H \leqslant K \in \mathscr{C}(G)} \sum_{\sigma \in \Delta^K} (-1)^{\dim(\sigma)} f_\sigma(K) = \sum_{\sigma \in \Delta^H} (-1)^{\dim(\sigma)} \sum_{\substack{H \leqslant K \leqslant G_\sigma \\ K \in \mathscr{C}(G)}} f_\sigma(K) = \sum_{\sigma \in \Delta^H} (-1)^{\dim(\sigma)}$$

by the defining property of f_σ. This last quantity is the Euler characteristic $\chi(\Delta^H)$, and if we assume condition (a) in the statement of Theorem A then it is 1 if $O_p(H) \neq 1$, $H \in \mathscr{C}(G)$. This completes the proof of Theorems A and A′ on the assumption that (a) holds.

We finish with the observation that (b) implies (a). If $H \in \mathscr{C}$ with $O_p(H) \neq 1$ then H has a subnormal series $H_1 \triangleleft H_2 \triangleleft H$ where H_1 has order p, H_2 is a p-group and H/H_2 is cyclic. Now Δ^{H_1} is acyclic by hypothesis, and hence $\mathbb{Z}/p\mathbb{Z}$-acyclic, thus by a theorem of Smith ([23], or VII, 10.5(b) in [3]) $\Delta^{H_2} = (\Delta^{H_1})^{H_2}$ is also $\mathbb{Z}/p\mathbb{Z}$-acyclic. Therefore Δ^{H_2} is $\mathbb{Q}$-acyclic and since Δ^H is the fixed points on Δ^{H_2} under the action of the cyclic group H/H_2, $\chi(\Delta^H) = \chi(\Delta^{H_2}) = 1$

by the Lefshetz trace formula. This completes the proof. The argument just indicated can also be found in Proposition 2 of [17].

It is interesting to compare the various rival conditions on Δ under which results like Theorem A are proved. A frequent condition in theorems proved by equivarient cohomology is that Δ^H is acyclic for all p-subgroups H of G. Evidently this implies condition (b), and in general it is a more stringent condition, as explained in [17]. In order to prove that Theorem A holds in the presence of (b), the condition (a) which we worked with in the actual proof has to be suitably weak. For example, if we replaced (a) by "Δ^H is $\mathbb{Z}/p\mathbb{Z}$-acyclic for all $H \in \mathscr{C}$" then we would not be able to deduce (a) from (b) (see [17]).

4. Proof of Theorem B and Corollary C

Quillen showed in [18] that if $\Delta = \mathscr{A}$, $\mathscr{S}$, or the Tits building of a Chevalley group and $H \leqslant G$ is a p-group then Δ^H is contractible, from which condition (b) of Theorem A follows. Clearly, for any simplicial complex arising from a poset of subgroups, G_σ will always fix σ pointwise since the vertices of σ are subgroups ordered by inclusion. Hence Theorem A applies to $\mathscr{A}$, $\mathscr{S}$, and buildings.

In fact a slight extension of Quillen's argument shows that when $\Delta = \mathscr{A}$ or $\mathscr{S}$ and $H \in \mathscr{S}$ with $O_p(H) \neq 1$ then Δ^H is contractible, as we now demonstrate in the case $\Delta = \mathscr{A}$. Write $H_p = O_p(H)$ and put $C = \Omega(\zeta(H_p))$, the largest central elementary abelian subgroup of H_p. Then $1 \neq C$ char $H_p \lhd H$, so $C \lhd H$. Let $A \in \mathscr{A}^H$ be any non-trivial elementary abelian p-subgroup normalized by H. Then A^{H_p} is non-trivial and is normalized by H since $H_p \lhd H$. Hence the assignments

$$A \to A^{H_p} \to A^{H_p} \cdot C \to C$$

take place inside $\mathscr{A}^H$ and give a contraction of $\mathscr{A}^H$ [3, p. 268].

Suppose now that Δ is the Tits building of a finite Chevalley group in characteristic p. We may take the parabolic subgroups of rank n to be the simplices of dimension n, and then if B is a Borel subgroup the parabolic subgroups containing B form a set of representatives for Δ/G. Since the isotropy group or stabilizer of a parabolic subgroup is its normalizer, and parabolic subgroups are self normalizing, we obtain Corollary C.

5. Structure of the complex of elementary abelian p-subgroups

We first establish the conclusion of Theorem E for an arbitrary graph with similar fixed point properties to $\mathscr{A}$, but under the hypothesis of connectivity. This was essentially proved by Oliver [17], and the situation is very similar to one analysed by Quillen [18]. We give an algebraic proof for the benefit of the reader.

5.1. THEOREM. *Let Δ be a finite connected graph on which G acts and let p be a fixed prime. Suppose that G acts without inversions (i.e. any element of G which fixed an edge, fixes its two end vertices) and that for every subgroup $P \leqslant G$ of order p, Δ^P is a non-empty tree. Then the p-adic completion $H_1(\Delta)_p$ is a projective $\mathbb{Z}_pG$-module. Furthermore, if $C_1 \xrightarrow{d} C_0$ is the chain complex of Δ then both of the short exact sequences*

$$0 \to H_1(\Delta)_p \to (C_1)_p \to \operatorname{Im}(d)_p \to 0$$

and

$$0 \to \operatorname{Im}(d)_p \to (C_0)_p \to \mathbb{Z}_p \to 0$$

are split as sequences of $\mathbb{Z}_pG$-modules.

Proof. Let Q by a Sylow p-subgroup of G. Since $\mathbb{Z}_pG$-modules are projective and morphisms split if and only if, respectively, they are projective and split on restriction to Q, it suffices to assume $G = Q$ is a p-group. Now for each $1 \neq H \leqslant G$, Δ^H is contractible. This is because if $P \lhd H$ is a normal subgroup of order p then Δ^P is a non-empty tree on which H acts and any finite group acting on a tree has a fixed point (Serre), so $\Delta^H = (\Delta^P)^H$ is also a non-empty tree. Thus $\bigcup_{1\neq H\leqslant G} \Delta^H$ has the homology of the poset of subgroups of G [3, IX, 11.2], and this is contractible since the poset has a maximal element [18, 1.5]. Hence the chain complex $D_i \to D_0$ of $\bigcup_{1\neq H\leqslant G} \Delta^H$ has the homology of a point, and since this subcomplex consists of those points where the action is not free we can write $C_i = D_i \oplus P_i$ for $i = 1, 2$ where the P_i are free modules. By examining the long exact sequence associated with the sequence of chain complexes $0 \to D. \to C. \to P. \to 0$ we see that $C.$ and $P.$ have the same reduced homology, so there is an exact sequence

$$0 \to H_1(\Delta) \to P_1 \to P_0 \to 0.$$

This splits since P_0 is projective, so $H_1(\Delta)$ is projective.

It remains to show that the short exact sequences split. The first one splits because $H_1(\Delta)_p$ is projective, and $\operatorname{Im}(d)_p$ is torsion free, being a subgroup of

$(C_0)_p$. For the second sequence, G has been assumed to be a p-group and we have seen that Δ^G is a non-empty tree. Thus at least one of the transitive permutation summands of $(C_0)_p$ is just $\mathbb{Z}_p$ and the restriction of the map $(C_0)_p \to \mathbb{Z}_p$ to this summand is the identity $\mathbb{Z}_p \to \mathbb{Z}_p$. The inverse of this map gives the desired splitting.

We now turn our attention to the connectivity of $\mathscr{A}$. Since $\mathscr{A}$ and $\mathscr{S}$ are homotopy equivalent, they have the same number of components and evidently the action of G on them is the same. Let $\tilde{\mathscr{A}}$ be a connected component of $\mathscr{A}$ and $\tilde{G}$ the set of elements of G which preserve $\tilde{\mathscr{A}}$, so the permutation representation of G on the components of $\mathscr{A}$ is the action on the cosets of $\tilde{G}$ in G, since G permutes the components transitively. We state the following for a group of p-rank $\leqslant 2$.

5.2. LEMMA. *Let* $\tilde{C}_. = \tilde{C}_1 \xrightarrow{\tilde{d}} \tilde{C}_0$ *be the chain complex of* $\tilde{\mathscr{A}}$. *Then*

(i) *the chain complex of* $\mathscr{A}$ *is* $C_. = \tilde{C}_. \uparrow_{\tilde{G}}^{G}$, *obtained by applying the induction functor to* $\tilde{C}_.$.

(ii) $H_r(\mathscr{A}) \cong H_r(\tilde{\mathscr{A}}) \uparrow_{\tilde{G}}^{G}$ *as* $\mathbb{Z}G$*-modules, for* $r = 1, 2$.

Proof. (i) $\mathscr{A}$ is (equivalent to) the induced G-poset $\coprod_{g \in \tilde{G}\backslash G} \tilde{\mathscr{A}}_g$ defined in an obvious way as the disjoint union of pairwise incomparable copies of $\tilde{\mathscr{A}}$ indexed by the cosets $\tilde{G}\backslash G$. Evidently passing to the associated chain complex commutes with the process of induction.

(ii) This is because $\mathbb{Z}G$ is projective, and hence flat as a $Z\tilde{G}$-module, so that taking homology commutes with tensoring.

Proof of Theorem E. Since the inclusion map $\mathscr{A} \hookrightarrow \mathscr{S}$ is a homotopy equivalence [18] it suffices to prove the result for $\mathscr{A}$. Projectivity of $H_1(\mathscr{A})$ follows from 5.1 and 5.2 (ii). Since

$$0 \to \ker(\tilde{d})_p \to (\tilde{C}_1)_p \to \operatorname{Im}(\tilde{d})_p \to 0$$

and

$$0 \to \operatorname{Im}(\tilde{d})_p \to (\tilde{C}_0)_p \to \mathbb{Z}_p \to 0$$

both split, and this is preserved under induction, the corresponding sequences for $\mathscr{A}$ split as well. It remains to show that $\tilde{H}_0(\mathscr{A})_p$ is projective. Now $H_0(\mathscr{A})_p$ is the permutation module $\mathbb{Z}_p \uparrow_{\tilde{G}}^{G}$. We will show that the augmentation map $\mathbb{Z}_p \uparrow_{\tilde{G}}^{G} \to \mathbb{Z}_p$ splits with a projective kernel. In [18] Quillen showed that $\tilde{G}$ is self-normalizing and is a strongly p-embedded subgroup of G, which means that for every $x \in G$ either $\tilde{G}^x \cap \tilde{G} = \tilde{G}$ or $\tilde{G}^x \cap \tilde{G}$ contains no elements of order p. It follows that $\tilde{G}$ contains a Sylow p-subgroup of G, and it suffices to show (by the theory of relative projectivity) that on restriction to $\tilde{G}$ the augmentation map splits with a projective kernel. By Mackey's theorem,

$$\mathbb{Z}_p \uparrow_{\tilde{G}}^{G} \downarrow_{\tilde{G}} = \bigoplus_{\tilde{G}x\tilde{G}} \mathbb{Z}_p \uparrow_{\tilde{G}^x \cap \tilde{G}}^{\tilde{G}}$$

where the sum is taken over double cosets, and for each double coset apart from $\tilde{G}$ itself, $|\tilde{G}^x \cap \tilde{G}|$ is prime to p. Hence all summands on the right are projective apart from a single copy of $\mathbb{Z}_p$ corresponding to the double coset $\tilde{G}$. On this summand the map $\mathbb{Z}_p \to \mathbb{Z}_p$ is the identity, so the inverse gives a splitting for the augmentation as a $\tilde{G}$-map, and the kernel is isomorphic to the sum of the remaining summands, which is projective.

The only extra complication in Theorem E over Theorem 5.1 is that $\mathscr{A}$ might not be connected. But then the stabilizer $\tilde{G}$ of a component of $\mathscr{A}$ is self-normalizing and strongly p-embedded, and as far as cohomology is concerned we might just as well work with $\tilde{G}$ as with G. This is shown in the next result.

5.3. PROPOSITION. *The restriction map* res: $\hat{H}^n(G, M)_p \to \hat{H}^n(\tilde{G}, M)_p$ *is an isomorphism.*

Proof. By [5] res is mono, and

$$\text{cores.res} = \sum_{\tilde{G}x\tilde{G}} c_x \cdot \text{res}_{\tilde{G}\cap\tilde{G}^x}^{\tilde{G}^x} \cdot \text{cores}_{\tilde{G}\cap\tilde{G}^x}^{\tilde{G}}$$

Every map on the right is zero on the p-part of cohomology, except for the summand with $\tilde{G} \times \tilde{G} = \tilde{G}$. This is because the other maps factor through $\hat{H}^n(\tilde{G}^x \cap \tilde{G}, M)$, and this has trivial p-part because $p \nmid |\tilde{G}^X \cap \tilde{G}|$ if $x \notin \tilde{G}$. Hence cores.res $= 1$ and res is epi on the p-part of cohomology.

There is a group-theoretical interpretation of 5.1 and 5.2 which we now mention. We return to the situation of 5.1 where G acts on a graph Δ without inversions. The quotient graph Δ/G acquires the structure of a graph of groups by choosing a connected lifting of Δ/G to Δ and assigning as vertex or edge groups of Δ/G the stabilizers in G of the liftings of the vertices or edges to Δ. For the application to $\mathscr{A}$ in case G has p-rank 2, we should replace $\mathscr{A}$ by a connected component $\tilde{\mathscr{A}}$ if necessary, and G by $\tilde{G}$. In any case, $\mathscr{A}/G \cong \tilde{\mathscr{A}}/\tilde{G}$. Let $\hat{G}$ denote the fundamental group of this graph of groups. There is a unique homomorphism

$\hat{G} \to G$ which is an isomorphism on corresponding vertex or edge stabilizers, and since Δ is connected it is surjective, by Bass-Serre theory [21]. The kernel N of this homomorphism may be identified as the fundamental group $\pi_1(\Delta)$, and it is the free group on a fundamental set of cycles in Δ. Thus N/N' is a free abelian group, and it is acted upon by $\hat{G}$ by means of conjugation within $\hat{G}$. Since N itself acts trivially on N/N', this free abelian group becomes a $\mathbb{Z}G$-module, and one sees that $N/N' \cong H_1(\Delta)$ as $\mathbb{Z}G$-modules. Thus we have:

5.4. COROLLARY. *With the hypotheses of* 5.1, $(N/N')_p$ *is a projective* $\mathbb{Z}_pG$*-module.*

This kind of situation was considered by Brown on p. 67 of [2]. There is a Mayer-Vietoris sequence giving the equivariant cohomology of G on Δ, and also a Mayer-Vietoris sequence for the cohomology of $\hat{G}$ [7]. These both have the form

$$\cdots \to \hat{H}^n(\hat{G}, M) \to \bigoplus_{v \in V} \hat{H}^n(G_v, M) \to \bigoplus_{e \in E} \hat{H}^n(G_e, M) \to \hat{H}^{n+1}(\hat{G}, M) \to \cdots$$

in the case of $\hat{G}$, or the same sequence with the $\hat{G}$ term replaced by $\hat{H}^n_G(\Delta; M)$ for equivariant cohomology. Here V and E are the vertex and edge sets of Δ, and we will always work with Farrell-Tate cohomology, denoted by $\hat{H}$. As Brown observed, it follows that for $\mathbb{Z}G$-modules M, inf: $\hat{H}^n(G, M)_p \to \hat{H}^n(\hat{G}, M)_p$ is an isomorphism, since $\hat{H}^n_G(\Delta; M)_p \cong \hat{H}^n(G, M)_p$ and because we have isomorphisms on the vertex and edge groups. Because of the information about $\hat{H}^n(G, M)_p$ in Theorem A we also obtain:

5.5. THEOREM. *Let M be a $\mathbb{Z}G$-module and $\hat{G}$ the fundamental group of the graph of groups Δ/G, with Δ as in* 5.1. *Then at the prime p the Mayer-Vietoris sequence for the cohomology of $\hat{G}$ with coefficients in M is the splice of split short exact sequences of the form*

$$0 \to \hat{H}^n(\hat{G}, M)_p \to \bigoplus_{v \in V} \hat{H}^n(G_v, M)_p \to \bigoplus_{e \in E} \hat{H}^n(G_e, M)_p \to 0$$

Proof. The isomorphism $\hat{H}^n(G, M)_p \cong \hat{H}^n(\hat{G}, M)_p$ and the formula of Theorem A show that the middle term in the above sequence is isomorphic to the direct sum of the two outer terms. By counting composition lengths it immediately follows that the above is a short exact sequence. It splits because of the following much more general theorem of Miyata:

THEOREM (Miyata [27]). *Let R be a ring with a Noetherian subring Z contained in the centre of R such that R is a finitely generated Z-module. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated R-modules. If $B \cong A \oplus C$ then the sequence splits.*

6. The connection with equivariant cohomology

The approach of equivariant cohomology is to obtain the cohomology of G in terms of the cohomology of the isotropy groups in an action of G on some suitable space. In our situation of G acting on Δ, if Δ^H is acyclic for all p-subgroups $1 \neq H \leq G$ then the p-part of the equivariant cohomology $\hat{H}^*_G(\Delta, M)$ is isomorphic to the p-part of $\hat{H}^*(G, M)$ [3, p. 292], and there is a spectral sequence whose E_1 page is

$$E_1^{r,s} = \bigoplus_{\sigma \in \Delta_r/G} \hat{H}^s(G_\sigma, M)_p \Rightarrow \hat{H}_G^{r+s}(\Delta, M)_p \tag{6.1}$$

[3, p. 173] where Δ_r is the set of simplices in dimension r. It would be interesting to prove Theorem A using this spectral sequence, but I have not been able to do so in general. The case where I can do it is the one treated in Theorem E, when Δ is a graph. Here the splitting of the differential and the projective homology immediately imply Theorem A′, and hence Theorem A, but Theorem E also demonstrates that the p-torsion part of the equivariant cohomology spectral sequence terminates at the E_2 page.

6.2. PROPOSITION. *Let G and Δ be as in Theorem E. The p-torsion equivariant cohomology spectral sequence* (6.1) *has $E_2 = E_\infty$ and the E_2 page is zero except on the fibre $E_2^{0,\cdot}$.*

Proof. The spectral sequence arises from the double complex $\hom_{\mathbb{Z}_pG}(\mathcal{P}, \hom_{\mathbb{Z}_p}(C.(\Delta)_p, M_p))$, where $\mathcal{P}$ is a $\mathbb{Z}_pG$ projective resolution of $\mathbb{Z}_p$. Since the reduced homology $\tilde{H}_r(\Delta)_p$ is always projective, it splits off from $C.(\Delta)_p$ and the remaining differential on $C.(\Delta)_p$ is then split (by Theorem E). It follows that $\hom_{\mathbb{Z}_p}(C.(\Delta)_p, M_p)$ also has projective homology and split differential so that when we take homology along the columns of the double complex the projective homology groups contribute nothing, and what remains is the E_1 page

$$\begin{array}{ccc} \vdots & & \vdots \\ \bigoplus_{\sigma\in\Delta_0/G} \hat{H}^1(G_\sigma, M)_p & \xrightarrow{d} & \bigoplus_{\sigma\in\Delta_1/G} \hat{H}^1(G_\sigma, M)_p \\ \bigoplus_{\sigma\in\Delta_0/G} \hat{H}^0(G_\sigma, M)_p & \xrightarrow{d} & \bigoplus_{\sigma\in\Delta_1/G} \hat{H}^0(G_\sigma, M)_p \end{array}$$

where each of the maps d is induced by the differential of $C.(\Delta)$ and is thus split epi. Hence E_2 is only non-zero on the fibre, and the spectral sequence stops there.

I am in fact able to show that the conclusion of Proposition 6.2 holds without restriction on the dimension of Δ, namely that the rows in the E_1 page of the above spectral sequence are all split acyclic, except at the left hand-end where the homology is $\hat{H}^*(G, M)_p$. I hope to return to this in another paper.

It is also just conceivable that the following question might always have an answer in the affirmative. This would immediately imply Theorems A and A′.

6.3. Let $\Delta = \mathcal{S}$ be the complex of p-subgroups of G and

$$C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0$$

be its chain complex. Is it true that for every r the reduced homology completed at p, $\tilde{H}_r(\Delta)_p$ is a projective $\mathbb{Z}_pG$-module, and the sequence

$$0 \to \ker (d_r)_p \to (C_r)_p \to \operatorname{Im} (d_r)_p \to 0$$

is split?

7. Cohomology of some specific groups

In this section we use Theorems A and D to give reduction formulae for the cohomology of certain specific groups in terms of the cohomology of their subgroups. As well as being valid for arbitrary $\mathbb{Z}G$-modules M we give more detailed results when M has the trivial action, and for this we consider the

Poincaré series of the cohomology ring with $\mathbb{F}_p$ coefficients:

$$P_G(t) = \sum_{n=0}^{\infty} t^n \cdot \dim H^n(G, \mathbb{F}_p)$$

The results for the Poincaré series when $p = 2$ are summarised in the following table

Group	$P_G(t)$
A_4	$\dfrac{1+t^3}{(1-t^3)(1-t^2)}$
D_n	$\dfrac{1}{(1-t)^2}$
Σ_4	$\dfrac{1+t^2}{(1-t)(1-t^3)}$
Σ_5	same as Σ_4
Σ_6	$\dfrac{1+t^3}{(1-t)(1-t^2)(1-t^3)}$
A_5	same as A_4
A_6	same as A_4
A_7	same as A_4
$PSL_2(q)$, q odd	same as A_4
$PSL_3(q)$, q odd	$\dfrac{1+t^5}{(1-t^3)(1-t^4)}$
M_{11}	$\dfrac{1+t^5}{(1-t^3)(1-t^4)}$
J_1	$\dfrac{(1+t^5)(1+t^6)}{(1-t^3)(1-t^4)(1-t^7)}$

The above groups will be taken in order through this section, where further formulae will appear with details of the calculations. From 7.4 onwards where the formulae become more complicated we will omit the coefficient module M from our notation; thus $H^n(G)_2$ will mean $H^n(G, M)_2$. The formulae still work for arbitrary modules M. Many of the above Poincaré series would be regarded as known, in particular the first two.

The series for A_4 may be obtained from the fact that $H^n(A_4, \mathbb{F}_2)$ is isomorphic under the restriction map to the fixed points $H^n(V, \mathbb{F}_2)^{C_3}$ where V is the Sylow 2-subgroup of A_4 [5]. Now $\bigoplus_{n=0}^{\infty} H^n(V, \mathbb{F}_2)$ is a polynomial ring with two generators in degree 1, and the action of C_3 is induced by that on $H^1(V, \mathbb{F}_2)$, which is the dual of the action of C_3 on V. The Poincaré series of the ring of invariants may now be computed using Molien's Theorem [24].

When $G = D_{2^m}$ is a dihedral 2-group the Poincaré series may be obtained from the description of the kernels in a minimal projective resolution of $\mathbb{F}_2$ provided by Butler and Shahzamanian [4]. From their description, the dimension of the maximal semisimple quotient of the nth kernel is $n+1$, and this is $\dim H^n(D_{2^m}, \mathbb{F}_2)$. Hence

$$\sum_{n=0}^{\infty} t^n \dim H^n(D_2m, \mathbb{F}_2) = 1 + 2t + 3t^2 + 4t^3 + \cdots = \frac{1}{(1-t)^2}.$$

We will treat the general dihedral group in 7.3.

From time to time we will use the following elementary but rather powerful observation.

7.1. LEMMA. (1) *Suppose* $1 \to N \to G \to Q \to 1$ *is a short exact sequence of groups and* $u_Q = \Sigma \lambda_K u_K$ *in* $A(Q)$ *where the sum is taken over various subgroups* $K \leq Q$. *Let* $\bar{K}$ *be the inverse image of* K *in* G. *Then* $u_G = \Sigma \lambda_K u_{\bar{K}}$ *in* $A(G)$, *and* $H^n(G, M)_p = \Sigma \lambda_K H^n(\bar{K}, M)_p$.

(2) *Suppose* $G = A \times B$ *and* $u_A = \Sigma \lambda_H u_H$ *in* $A(A)$, $u_B = \Sigma \lambda_K u_K$ *in* $A(B)$, *for subgroups* $H \leq A$ *and* $K \leq B$. *Then* $u_G = \Sigma \lambda_H \lambda_K u_{H \times K}$ *and* $H^n(G, M)_p = \Sigma \lambda_H \lambda_K H^n(H \times K, M)_p$.

Proof. The cohomology formulae follow from those for permutation modules as explained in §2.

(1) We regard $u_Q = \Sigma \lambda_K u_K$ as an equation of $\mathbb{Z}_p G$-modules via the homomorphism $G \to Q$.
(2) $u_G = u_A \otimes u_B = \Sigma \lambda_H \lambda_K u_H \otimes u_K = \Sigma \lambda_H \lambda_K u_{H \times K}$.

7.2. $G = \Sigma_4$, $p = 2$

We apply Theorem D with $\mathscr{X} = \mathscr{C}$, the subgroups of Σ_4 which are cyclic (mod 2). The maximal members of $\mathscr{C}$ are A_4 and three copies of D_8. Every pair of these intersects in the four group V which is normal in Σ_4 so these are the only subgroups which arise as intersections of maximal members. As explained in §2, the function f of Theorem D is only non-zero on these subgroups, and we have $f(A_4)=f(D_8)=1$, $f(V)=-3$. Theorem D becomes

$$H^n(\Sigma_4, M)_2=\tfrac{1}{2}H^n(A_4, M)_2+H^n(D_8, M)_2-\tfrac{1}{2}H^n(V, M)_2$$

where $V \lhd \Sigma_4$. The Poincaré series is now computed as

$$P_{\Sigma_4}(t)=\tfrac{1}{2}P_{A_4}(t)+P_{D_8}(t)-\tfrac{1}{2}P_V(t).$$

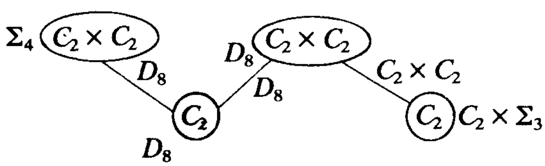
7.3. $G=D_{2^rm}$ *where m is odd*, $p=2$

The procedure is the same as for Σ_4. Maximal members of $\mathscr{C}: C_{2^{r-1}m}$ and D_{2^r} (m copies). These intersect only in the subgroup $C_{2^{r-1}}$. Therefore $f(C_{2^{r-1}m})=f(D_{2^r})=1$, $f(C_{2^{r-1}})=-m$. The indices of these subgroups in their normalizers are, respectively, 2, 1 and $2m$. Hence by Theorem D

$$H^n(D_{2^rm}, M)_2=H^n(D_{2^r}, M)+\tfrac{1}{2}[H^n(C_{2^{r-1}m}, M)_2-H^n(C_{2^{r-1}}, M)]$$

7.4. $G=\Sigma_5$, $p=2$

We apply Theorem A. The quotient graph $\mathscr{A}/G$ may be represented as follows:



This means that there are two conjugacy classes of subgroups of type $C_2\times C_2$, and two of type C_2. The normalizers of the corresponding subgroups are Σ_4, D_8, D_8 and $C_2\times\Sigma_3$. For each conjugacy class of subgroups C_2 and $C_2\times C_2$ there is at most one orbit of edges $C_2\subseteq C_2\times C_2$, and the edge stabilizers are shown adjacent to the edges. Reading from left to right, typical representatives of the $C_2\times C_2$ subgroups are $\langle(12)(34), (13)(24)\rangle$, $\langle(12),(34)\rangle$; and of the C_2 subgroups are $\langle(12)(34)\rangle$ and $\langle(12)\rangle$. All of the D_8 subgroups shown are the same. It follows by Theorem A that

$$\begin{aligned} H^n(G)_2 &= H^n(\Sigma_4)_2+2H^n(D_8)+H^n(C_2\times\Sigma_3)_2-2H^n(D_8)-H^n(C_2\times C_2)\\ &= H^n(\Sigma_4)_2+H^n(C_2\times\Sigma_3)_2-H^n(C_2\times C_2)_2. \end{aligned}$$

We are now omitting the coefficient module M from our notation. This equation holds for arbitrary coefficients M. Some simplification of the last two terms is possible using Theorem D. We know by Theorem D′ that

$$u_{\Sigma_3} = u_{C_2} + \tfrac{1}{2}(u_{C_3} - u_1).$$

So that by Lemma 7.1, $u_{C_2\times\Sigma_3} = u_{C_2\times C_2} + \frac{1}{2}(u_{C_2\times C_3} - u_{C_2})$. Hence

$$\begin{aligned} H^n(G)_2 &= H^n(\Sigma_4)_2 + H^n(C_2\times C_2) + \tfrac{1}{2}[H^n(C_2\times C_3)_2 - H^n(C_2)] - H^n(C_2\times C_2)\\ &= H^n(\Sigma_4)_2 + \tfrac{1}{2}[H^n(C_2\times C_3)_2 - H^n(C_2)]. \end{aligned}$$

By the Künneth formula, the $\mathbb{F}_2$-cohomology ring for $C_2\times C_3$ has the same Poincaré series as for C_2, and so $P_G(t) = P_{\Sigma_4}(t)$.

7.5. $G = \Sigma_6, p = 2$

To describe $\mathscr{A}/G$ is rather complicated, since there are 3 conjugacy classes of C_2 subgroups, 5 classes of $C_2\times C_2$, and 2 classes of $C_2\times C_2\times C_2$, and indeed, this is not the best approach. Up to conjugacy the maximal 2-local subgroups have the form $\Sigma_4\times C_2$ where Σ_4 permutes four of the letters, and $N_G(\langle(12), (34), (56)\rangle)$. This latter group has the structure $C_2\times C_2\times C_2 \lhd \Sigma_3$, since the three transpositions shown are the only ones in the group they generate, and this set of three elements is preserved. We apply Theorem D with $\mathscr{X} = \{\text{all 2-local subgroups}\}$, i.e. subgroups of $\Sigma_4\times C_2$, $C_2\times C_2\times C_2 \lhd \Sigma_3$, and their conjugates. These are the maximal elements of $\mathscr{X}$. A calculation shows that the possible intersections of these have the form

1. $\Sigma_4\times C_2$
2. $C_2\times C_2\times C_2 \lhd \Sigma_3$
3. Σ_3 permuting 3 of the letters
4. $D_8\times C_2$ with D_8 permuting 4 letters
5. $\langle(12), (34), (56)\rangle$
6. $\langle(12)(34), (13)(24), (56)\rangle$
7. $\langle(12)\rangle$

Intersections which have odd order have been omitted. Taking the rows and columns to correspond to these subgroups in the above order, the matrix C discussed in section 2 is

$$C = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 4 & 3 & 1 & & & & \\ 1 & 1 & 0 & 1 & & & \\ 3 & 1 & 0 & 3 & 1 & & \\ 1 & 3 & 0 & 3 & 0 & 1 & \\ 7 & 3 & 4 & 9 & 3 & 1 & 1 \end{bmatrix}$$

solving the equation $C \cdot f = (1, \ldots, 1)'$ gives $f = (1\ 1\ -6\ -1\ 0\ 0\ 24)'$, and the vector with entries $f(K)/|N_G(K):K|$ is $(1\ 1\ -1\ -1\ 0\ 0\ 1)'$. Therefore by Theorem D,

$$H^n(\Sigma_6)_2 = H^n(\Sigma_4 \times C_2)_2 + H^n(C_2 \times C_2 \times C_2 \rtimes \Sigma_3)_2 - H^n(D_8 \times C_2) - H^n(\Sigma_3)_2 + H^n(C_2).$$

Note that in the last two terms, C_2 may be chosen to be a subgroup of Σ_3, and since $H^n(\Sigma_3)_2 = H^n(C_2)$, the last two terms cancel. Some further simplification of the remaining terms is possible, but we must take care to distinguish non-conjugate subgroups which are abstractly isomorphic. Since $u_{\Sigma_4} = u_{D_8} + \frac{1}{2}(u_{A_4} - u_{C_2 \times C_2})$, by Lemma 7.1 we obtain $H^n(\Sigma_4 \times C_2)_2 = H^n(D_8 \times C_2) + \frac{1}{2}[H^n(A_4 \times C_2)_2 - H^n(C_2 \times C_2 \times C_2)]$, and since $u_{\Sigma_3} = u_{C_2} + \frac{1}{2}(u_{C_3} - u_1)$ we have

$$H^n(C_2 \times C_2 \times C_2 \rtimes \Sigma_3)_2 = H^n(D_8 \times C_2) + \tfrac{1}{2}[H^n(C_2 \times C_2 \times C_2 \rtimes C_3)_2 - H^n(C_2 \times C_2 \times C_2)]$$

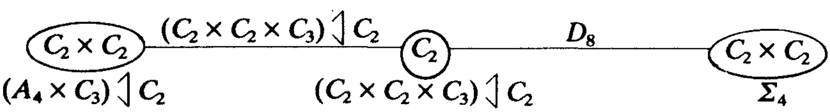
Substituting this into the formula for $H^n(\Sigma_6)_2$ gives

$$H^n(\Sigma_6)_2 = H^n(D_8 \times C_2) + \tfrac{1}{2}[H^n(A_4 \times C_2)_2 - H^n(C_2 \times C_2 \times C_2) + H^n(C_2 \times C_2 \times C_2 \rtimes C_3)_2 - H^n(C_2 \times C_2 \times C_2)]$$

In this formula, one of the groups $C_2 \times C_2 \times C_2$ is $\langle (12), (34), (56) \rangle$ while the other is $\langle (12)(34), (13)(24), (56) \rangle$. Extending each of these groups by the 3-cycles (135)(246) and (123), respectively, we obtain the groups denoted by $C_2 \times C_2 \times C_2 \rtimes C_3$ and $A_4 \times C_2$, which are abstractly isomorphic. The formula holds for arbitrary coefficient modules M.

7.6. $G = A_5, A_6$ or A_7, $p = 2$

When $G = A_5$, a Sylow 2-subgroup P is a TI set and so $H^n(G, M)_2 \cong H^n(N_G(P), M)_2$ by [5]. If $G = A_6$ Then $G \cong PSL(2, 9)$, and this will be covered by the discussion of the groups $PSL(2, q)$. Now let $G = A_7$. We apply Theorem A. $\mathcal{A}/G$ is



with vertex and edge stabilizers of $\mathcal{A}$ as shown.

A copy of the group $(C_2 \times C_2 \times C_3) \rtimes C_2$ is generated by the elements

$$(12)(34),\ (13)(24),\ (567),\ (12)(56),$$

and representatives of the two conjugacy classes of four-groups are $V_1 = \langle (12)(34),\ (13)(24) \rangle$ and $V_2 = \langle (12)(34),\ (12)(56) \rangle$. Hence $H^n(A_7)_2 = H^n(\Sigma_4)_2 + H^n((A_4 \times C_3) \rtimes C_2)_2 - H^n(D_8)_2$.

We may reduce this further by applying the formula already obtained for Σ_4, and applying Theorem D to $(A_4 \times C_3) \rtimes C_2$, as follows. In Theorem D we take $\mathscr{X}$ to be all subgroups of $A_4 \times C_3$ and D_8, and conjugates of these. Then $\mathscr{X} \supseteq \mathscr{C}$, and the maximal members of $\mathscr{X}$ are $A_4 \times C_3$ and 9 copies of D_8, any two of which intersect in exactly $C_2 \times C_2$. Hence $f(A_4 \times C_3) = f(D_8) = 1$, $f(C_2 \times C_2) = -9$, and the indices of these subgroups in thein normalizers are 2, 1 and 18 respectively. By Theorem D,

$$H^n((A_4 \times C_3) \rtimes C_2)_2 = H^n(D_8)_2 + \tfrac{1}{2}H^n(A_4 \times C_3)_2 - \tfrac{1}{2}H^n(C_2 \times C_2)_2.$$

We substitute this and the formula for Σ_4 into the formula given above for A_7 to obtain

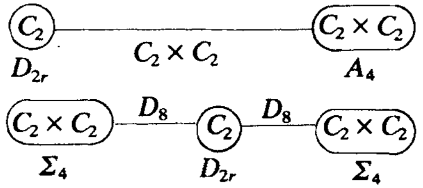
$$H^n(A_7)_2 = \tfrac{1}{2}H^n(A_4)_2 + \tfrac{1}{2}H^n(A_4 \times C_3)_2 + H^n(D_8) - \tfrac{1}{2}H^n(V_1) - \tfrac{1}{2}H^n(V_2).$$

The first group A_4 here normalizes the second four-group V_2, and $A_4 \times C_3$ acts as A_4 on $\{1, 2, 3, 4\}$ and as C_3 on $\{5, 6, 7\}$. Since A_4 and $A_4 \times C_3$ have the same cohomology with $\mathbb{F}_2$ coefficients (by the Künneth formula), the Poincaré series is

$$P_{A_7}(t) = P_{A_4}(t) + P_{D_8}(t) - P_{C_2 \times C_2}(t) = P_{A_4}(t).$$

7.7. $G = PSL(2, q)$, q odd, $p = 2$.

The structure of $\mathscr{A}/G$ is as follows:



if $q \equiv \pm 3 \pmod 8$

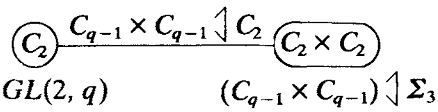
if $q \equiv \pm 1 \pmod 8$

(see [9]). In these diagrams r is the integer such that D_{2r} is the centralizer of an involution.

$$H^n(G)_2 = H^n(A_4)_2 + H^n(D_{2r})_2 - H^n(C_2 \times C_2)_2 \quad \text{if} \quad q \equiv \pm 3 \pmod 8$$
$$H^n(G)_2 = H^n(\Sigma_4)_2 + H^n(\Sigma_4)_2 + H^n(D_{2r})_2 - H^n(D_8) - H^n(D_8) \quad \text{if} \quad q \equiv \pm 1 \pmod 8).$$

We repeat groups according to the different conjugacy classes.

7.8. $G = PSL(3, q)$, q odd, $p = 2$
The structure of $\mathscr{A}/G$ is



(see [1]). Hence

$$\begin{aligned} &H^n(PSL(3, q))_2 \\ &\quad = H^n(GL(2, q))_2 + H^n(C_{q-1} \times C_{q-1} \rtimes \Sigma_3)_2 - H^n(C_{q-1} \times C_{q-1} \rtimes C_2)_2. \end{aligned}$$

We may reduce the middle term on the right using Theorem B. With $G = (C_{q-1} \times C_{q-1}) \rtimes \Sigma_3$, take $\mathscr{X}$ to consist of all subgroups of $(C_{q-1} \times C_{q-1}) \rtimes C_3$, $(C_{q-1} \times C_{q-1}) \rtimes C_2$ and their conjugates. These are the maximal elements of $\mathscr{X}$ and any two of them intersect in $C_{q-1} \times C_{q-1}$. Hence $f((C_{q-1} \times C_{q-1}) \rtimes C_3) = 1 = f((C_{q-1} \times C_{q-1}) \rtimes C_2)$, $f(C_{q-1} \times C_{q-1}) = -3$.

$$\begin{aligned} H^n((C_{q-1} \times C_{q-1}) \rtimes \Sigma_3)_2 &= \tfrac{1}{2} H^n((C_{q-1} \times C_{q-1}) \rtimes C_3)_2 \\ &\quad + H^n((C_{q-1} \times C_{q-1}) \rtimes C_2)_2 - \tfrac{1}{2} H^n(C_{q-1} \times C_{q-1})_2 \end{aligned}$$

Therefore

$$\begin{aligned} &H^n(PSL(3, q))_2 \\ &\quad = H^n(GL(2, q))_2 + \tfrac{1}{2}[H^n(C_{q-1} \times C_{q-1} \rtimes C_3)_2 - H^n(C_{q-1} \times C_{q-1})_2] \end{aligned}$$

We can use this to determine the Poincaré series for $PSL(3, q)$ over $\mathbb{F}_2$. By work of Quillen,

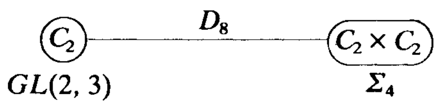
$$P_{GL(2,q)}(t) = \frac{1 + t^3}{(1 - t)(1 - t^4)}$$

(see [19] or [10]). The cohomology ring of $C_{q-1} \times C_{q-1}$ is a polynomial ring in two variables, and that of $C_{q-1} \times C_{q-1} \lhd C_3$ is the fixed points under the action of C_3 on this ring [5]. $C_{q-1} \times C_{q-1} \lhd C_3$ thus has the same Poincaré series as A_4, and

$$P_{PSL(3,q)}(t) = \frac{1+t^3}{(1-t)(1-t^4)} + \frac{1}{2}\left[\frac{1+t^3}{(1-t^3)(1-t^2)} - \frac{1}{(1-t)^2}\right]$$
$$= \frac{1+t^5}{(1-t^3)(1-t^4)}$$

7.9. *M_{11} has the same 2-local structure as $PSL(3,3)$*

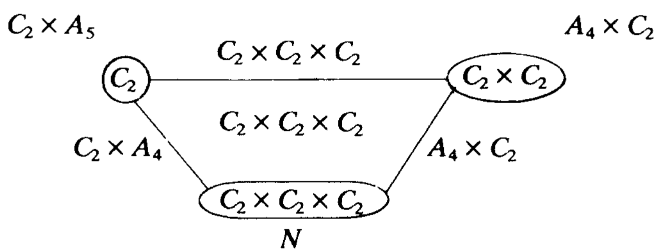
The graph $\mathscr{A}/G$ for M_{11} is



7.10. $G = J_1$, $p = 2$

Because Sylow 2-subgroups are abelian the 2-cohomology with trivial coefficients may be computed using Swan's theorem, the details being given in Chapman [6]. Chapman gives the expression for $P_G(t)$, and this may be shown to be correct using Molien's Theorem. The normalizer of a Sylow 2-subgroup has the structure $N = (C_2 \times C_2 \times C_2) \lhd (C_7 \lhd C_3)$, and for trivial coefficients $H^n(G, M)_2 \cong H^n(N, M)_2$. This result should be modified for arbitrary coefficient modules M, as we shall see.

By [15], $\mathscr{A}/G$ has the structure



with vertex and edge stabilizers as shown. Therefore by Theorem A, and after some cancellation,

$$H^n(G)_2 = H^n(N)_2 + H^n(C_2 \times A_5)_2 - H^n(C_2 \times A_4)_2.$$

Some simplification of the last two terms is possible. By Theorem D,

$$u_{A_5} = u_{A_4} + \tfrac{1}{2}(u_{C_5} - u_{C_3})$$

and so by Lemma 6.1, $u_{C_2\times A_5} = u_{C_2\times A_4} + \tfrac{1}{2}(u_{C_2\times C_5} - u_{C_2\times C_3})$. Hence

$$H^n(G)_2 = H^n(N)_2 + \tfrac{1}{2}(H^n(C_2\times C_5)_2 - H^n(C_2\times C_3)_2).$$

Some reduction of $H^n(N)_2$ is also possible with Theorem D, but we do not give this.

8. Euler characteristic formulae

It was proved by K. S. Brown (see [2]) that the Euler characteristic of $\mathscr{A}$ satisfies

$$\chi(\mathscr{A}) \equiv 1 \ (\mathrm{mod}\ |G|_p).$$

This was significant in his investigation of the Euler characteristic of G, and was reproved by Quillen [18] and Gluck [11]. In this section we show that our own Theorem A′ contains this congruence, and prove some other formulae of a similar nature. We work in the generality of a group acting on a simplicial complex so that condition (a) of Theorem A is satisfied, and this includes the cases $\mathscr{A}$ and $\mathscr{S}$. The condition is: (a) for all $H \in \mathscr{C}$ with $p \,\|\, |H|$, $\chi(\Delta^H) = 1$.

8.1. THEOREM. *Let G act on the simplicial complex Δ so that (a) holds. Then*

$$\chi(\Delta) \equiv 1 \ (\mathrm{mod}\ |G|_p).$$

Proof. Take ranks of both sides in Theorem A′. We obtain

$$1 \equiv \sum_{\sigma\in\Delta/G} (-1)^{\dim(\sigma)} |G:G_\sigma| = \chi(\Delta) \ (\mathrm{mod}\ |G|_p),$$

the congruence arising because every finite rank projective $\mathbb{Z}_pG$-module has rank divisible by $|G|_p$.

In the next result we impose the further condition that all isotropy groups have order divisible by p. This is satisfied when $\Delta = \mathscr{A}$ or $\mathscr{S}$, since if $\sigma = E_0 < \cdots < E_n$ is a simplex then E_0 is a non-identity p-group with $E_0 \subseteq G_\sigma$.

8.2. PROPOSITION. *Let G act on the simplicial complex Δ so that (a) holds and for all $\sigma \in \Delta$, $p \| |G_0|$.*

(i) *Let Δ/G denote the quotient complex of Δ by the action of G. Then $\chi(\Delta/G) = 1$.*

(ii) $|G|_p = \prod_{\sigma\in\Delta/G} |G_\sigma|_p^{(-1)^{\dim\sigma}}$ *and* $|G/G'|_p = \prod_{\sigma\in\Delta/G} |G_\sigma/G'_\sigma|_p^{(-1)^{\dim\sigma}}$

Proof. (i) Recall from [5] that $\hat{H}^0(G, \mathbb{Z})$ is cyclic of order $|G|$. Substituting this into Theorem A we obtain

$$C_{|G|_p} = \sum_{\sigma\in\Delta/G} (-1)^{\dim\sigma} C_{|G_\sigma|_p}.$$

We may take the negative terms over to the left hand side and take the rank of both sides as p-groups to obtain

$$1 = \sum_{\sigma\in\Delta/G} (-1)^{\dim\sigma}$$

after returning the negative terms to the right hand side. This is now precisely $\chi(\Delta/G)$.

(ii) Instead of taking the rank of both sides in (i), take the order of both sides to obtain the first formula. The second formula follows in a similar way using the (co)homology group $\hat{H}^{-2}(G, \mathbb{Z}) = H_1(G, \mathbb{Z}) \cong G/G'$. Evidently a formula of the type in 8.2 (ii) will hold for the order of any cohomology group, for example the Schur multiplier.

8.3. COROLLARY. *If G has p-rank 2 then $\mathcal{A}/G$ is a tree.*

Proof. $\mathcal{A}/G$ is a connected graph with Euler characteristic 1.

The last result is a statement about the p-local structure of groups of p-rank 2. For example, it implies that in such a group, if x and y are commuting elements of order p with $\langle x, y\rangle \cong C_p \times C_p$ and $\langle x\rangle$ is conjugate to $\langle y\rangle$ in G, then $\langle x\rangle$ is conjugate to $\langle y\rangle$ in $N_G(\langle x, y\rangle)$. This is because in $\mathcal{A}$ there are edges $\langle x\rangle \subseteq \langle x, y\rangle$ and $\langle y\rangle \subseteq \langle x, y\rangle$ whose end points fuse in $\mathcal{A}/G$. Because $\mathcal{A}/G$ is a tree, there is only one edge between these two vertices of $\mathcal{A}/G$, so the above two edges of $\mathcal{A}$ are conjugate. This means there is an element $g \in G$ with $\langle x\rangle^g = \langle y\rangle$ and $\langle x, y\rangle^g = \langle x, y\rangle$. It seems, however, that the overall p-local information conveyed by 8.3 is of a more subtle nature than this. It is interesting also that Corollary 8.3 retains some force even when G is a p-group, in contrast to many of the results in this paper.

The approach adopted in 8.2 can also be applied to Theorem D to yield a number of similar results. We use the notation of Theorem D.

8.4. PROPOSITION. *Let $\mathscr{X}$ be any class of subgroups of G which is closed under conjugation and taking subgroups, and which contains the class $\mathscr{C}$ of subgroups which are cyclic mod p.*

(i) $\sum_{H\in\mathscr{X}} f(H)=1; \qquad \sum_{\substack{H\in\mathscr{X}\\ p\,\|\,|H|}} f(H)\equiv 1 \ (\mathrm{mod}\ |G|_p)$

(ii) $\sum_{H\in\mathscr{X}} \frac{f(H)}{|G:H|}=1; \qquad \sum_{\substack{H\in\mathscr{X}\\ p\,\|\,|H|}} \frac{f(H)}{|G:H|}=1$

(iii) $|G|_p=\prod_{\substack{H\in\mathscr{X}\\ p\,\|\,|H|}} |H|_p^{(f(H)/|G:H|)}$

Proof. (i) The equation is really included for completeness, because it is one of the defining equations for f. We may also verify it by taking ranks of both sides of the equation in Theorem D′. This is how we prove the second formula, except that on omitting the terms for which $p \nmid |H|$ we obtain a congruence mod $|G|_p$, since for such H, rank u_H is divisible by $|G|_p$.

(ii) The formula in Theorem D′ is equivalent to an isomorphism between two direct sums of modules. Taking fixed points and then ranks of both sides gives the first equation. For the second we use the expression in Theorem D for $\hat{H}^\circ(G, \mathbb{Z})_p \cong C_{|G|_p}$ and take the rank of each side as a p-group.

(iii) Follows by taking the order of both sides of the equation for $\hat{H}^\circ(G, \mathbb{Z})_p$. A similar formula holds for the order of any other cohomology group.

Remark. The sums in 8.4 are often more easily evaluated if taken over a set $\mathscr{X}^*$ of representatives of conjugacy classes of subgroups in $\mathscr{X}$, and factors $|G:N_G(H)|$ are introduced. These identities have a use as a check on the accuracy of one's calculation of the values of the function f, computed, for example, as described in §2.

REFERENCES

[1] J. L. ALPERIN, R. BRAUER and D. GORENSTEIN, *Finite Groups with quasi dihedral and wreathed Sylow 2-subgroups*, Trans. Amer. Math. Soc. *151* (1970), 1–261.
[2] K. S. BROWN, *Groups of virtually finite dimension*, in: ed. C. T. C. Wall, *Homological Group Theory*, L.M.S. Lecture Notes no. 36, Cambridge Univ. Press 1979.
[3] K. S. BROWN, *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer 1982.

[4] M. C. R. BUTLER and M. SHAHZAMANIAN, *The construction of almost split sequences, III: modules over two classes of tame local algebras*, Math. Annalen *247*, 111–122 (1980).
[5] H. CARTAN and S. EILENBERG, *Homological algebra*, Princeton Univ. Press 1956.
[6] G. R. CHAPMAN, *Generators and relations for the cohomology ring of Janko's first group in the first twenty one dimensions*, in: ed. C. M. Campbell and E. F. Robertson, Groups—St. Andrews 1981, L.M.S. Lecture Notes no. 71, Cambridge Univ. Press 1982.
[7] I. M. CHISWELL, *Exact sequences associated with a graph of groups*, J. Pure Appl. Algebra *8* (1976), 63–74.
[8] S. B. CONLON, *Decompositions induced from the Burnside algebra*, J. Algebra *10* (1968), 102–122.
[9] L. E. DICKSON, *Linear groups with an exposition of the Galois field theory*, Dover (New York) 1958.
[10] Z. FIEDOROWICZ and S. PRIDDY, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Lecture Notes in Math. 674, Springer 1978.
[11] D. GLUCK, *Idempotent formula for the Burnside algebra with applications to the p-subgroup simplicial complex*, *Ill.* J. Math. *25* (1981), 63–67.
[12] K. W. GRUENBERG, *Cohomological Topics in Group Theory*, Lecture Notes in Math. 143, Springer (Berlin–Heidelberg–New York) 1970.
[13] P. HALL, *The Eulerian functions of a group*, Quart. J. Math. Oxford Series 7 (1936), 134–151.
[14] D. F. HOLT, *On the local control of Schur multipliers*, Quart. J. Math. Oxford Series 28 (1977), 495–508.
[15] Z. JANKO, *A new finite simple group with abelian Sylow 2-subgroups*, J. Algebra *3* (1966), 147–186.
[16] M. MIYAMOTO, *An affirmative answer to Glauberman's conjecture*, Pacific J. Math. *102* (1982), 89–105.
[17] R. OLIVER, *Fixed-point sets of group actions on finite acyclic complexes*, Comment. Math. Helvetici *50* (1975), 155–177.
[18] D. QUILLEN, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Advances in Math. *28* (1978), 101–128.
[19] D. QUILLEN, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) *96* (1972), 552–586.
[20] K. W. ROGGENKAMP and L. SCOTT, *Hecke actions on Picard groups*, J. Pure Appl. Algebra *26* (1982), 85–100.
[21] J.-P. SERRE, *Trees*, Springer (Berlin-Heidelberg-New York) 1982.
[22] L. SOLOMON, *The Burnside algebra of a finite group*, J. Combinatorial Theory *2* (1967), 603–615.
[23] P. A. SMITH, *Fixed points of periodic transformations*, Amer. Math. Soc. Coll. Pub. *27* (1942), 350–373.
[24] T. A. SPRINGER, *Invariant Theory*, Lecture Notes in Math. 585, Springer (Berlin-Heidelberg-New York) 1977.
[25] T. YOSHIDA, *Idempotents of Burnside rings and Dress induction theorem*, J. Algebra *80* (1983), 90–105.
[26] C. W. CURTIS and I. REINER, *Methods in Representation Theory*, vol. 1, J. Wiley and Sons 1981.
[27] T. MIYATA, *Note on direct summands of modules*, J. Math. Kyoto Univ. 7 (1967), 65–69.

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