Math 1572 **Spring, 2008**
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Practice Final

These are some sample questions similar to what will be on Final Exam.

1. Integrate

   (a)
$$\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$$

   (b)
$$\int e^{\ln\sqrt{x}}\,dx$$

   (c)
$$\int \frac{d\theta}{\sin\theta\cos\theta}$$

   (d)
$$\int_{-\infty}^{0} xe^{3x}\,dx$$

2. Determine the center of mass of the two (disjoint) regions of the plane, A and B, where A is the rectangle with sides $x=1$, $x=3$, $y=1$ and $y=-2$, and B is the triangle with sides $x=-2$, $y=0$, and $x=-y$. (This is not the two centers of mass for the two regions individually, but the single center of the mass of the two regions taken together.)

3. Integrate
$$\int \frac{16x^3}{4x^2-4x+1}\,dx$$

4. Find the limits

   (a)
$$\lim_{x\to\pi/3} \frac{\cos x - 0.5}{x-\pi/3}$$

   (b)
$$\lim_{x\to\infty} x\tan\frac{1}{x}$$

   (c)
$$\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$$

5. Determine which of the sequences below converges or not. If it converges, find the limit.

   (a)
$$\lim_{n\to\infty} \frac{n}{2^n}$$

(b)

$$\lim_{n\to\infty}\left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right)$$

(c)

$$\lim_{n\to\infty}\left(\ln n-\ln(n+1)\right)$$

6. Find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} C_n x^n\,,$$

where C_n is the n-th Catalan number,

$$C_n=\binom{2n}{n}/(n+1)\,.$$

Also, determine convergence at the endpoints.

7. Compute the Taylor polynomial (about 0) of degree 3 for the function

$$f(x)=\frac{\cos x}{\sqrt{1-x}}$$

8. Determine which of the following series converges or not. If it converges, determine whether the convergence is absolute or conditional.

(a)

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$$

(b)

$$\sum_{n=3}^{\infty}\frac{n+1}{n!}$$

(c)

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt[3]{n^2+n+1}}{n+\sqrt{n}}$$

9. Determine the radius of convergence for each power series. Also determine convergence at the end points.

(a)

$$\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$$

(b)

$$\sum_{n=1}^{\infty}\frac{n^3x^n}{n+1}$$

10. Find the values of x ($x\neq -1/2$) for which the series

$$\sum_{n=1}^{\infty}\frac{nx^n}{(n+1)(2x+1)^n}$$

converges, for which it converges absolutely, and for which it diverges.

11. Let

$$f(x) = \sum_{n=0}^{\infty} (-2)^n (n+1)\, x^n \,.$$

(a) Find the radius of convergence for this power series.

(b) Identify the function that this power series represents.

12. Determine the Cartesian equation for the curve given parametrically by

$$x = 2 + 1/t \text{ and } y = 2 - t, \qquad t > 0$$

13. Convert the Cartesian equation

$$xy = 1$$

to polar. Find the area enclosed by the curve and the two rays $\theta = \pi/6$ and $\theta = \pi/3$.

14. A particle is moving according to the equations

$$x(t) = \cosh t$$
$$y(t) = t$$

Compute the speed, curvature, and the tangential and normal components of acceleration.

15. Find the equation of the plane which is perpendicular to the plane given by

$$2x + z = 2$$

and contains the line given by

$$\frac{x-2}{3} = \frac{y+1}{2} = z - 1$$

16. Find the distance between the line described as the intersection of the planes

$$-x - y = 2$$

and

$$x + 3z = -1$$

and the point $(-1, 0, 1)$.

17. Define an ellipsoid by the equation

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

(a) Express the area of a cross-section cut of this ellipsoid by the plane $z = c$ as a function of c. (Recall that the area of an ellipse is πab where a and b are the semiaxes.)

(b) Using slices perpendicular to the z-axis, find the volume of this ellipsoid.

18. Let

$$\mathbf{R}(t) = 2\sqrt{t}\cos t\,\mathbf{i} + 3\sqrt{t}\sin t\,\mathbf{j} + \sqrt{1-t}\,\mathbf{k} \qquad 0 \le t \le 1$$

This parametric equation describes a curve in 3-space. It lies on a quadric surface. Find the equation for this surface and identify it.

Solutions

1. (a) Substitute $u = e^t$ to get

$$\int \frac{du}{u^2 + 3u + 2}$$

The denominator factors into $(u+2)(u+1)$. Use partial fractions to get

$$\ln \frac{e^t + 1}{e^t + 2}$$

(b) The integrand is simply $\sqrt{x}$, so the integral is $2x^{3/2}/3$.

(c) Use $\sin\theta\cos\theta = (1/2)\sin 2\theta$. Then integrate the resulting $2\csc 2\theta$ to get $-\ln(\csc 2\theta + \cot 2\theta)$.

(d) Integrate by parts to get the indefinite integral equal to $(x/3 - 1/9)e^{3x}$. Evaluating this between z and 0, and taking the limit as $z \to -\infty$ gives $-1/9$.

2. First, find the moment about the x-axis. For region A, this will be

$$\int_{-2}^{1} 2y\,dy = -3$$

For the region B, this will be

$$\int_{0}^{2} (2-y)y\,dy = 4/3$$

So the moment will be the some of these (since the integral of the sum is the sum of the integrals), or $-5/3$.

Next, find the moment about the y-axis. For the region A, this will be

$$\int_{1}^{3} 3x\,dx = 12$$

For the region B it will be

$$\int_{-2}^{0} -x^2\,dx = -8/3$$

Adding these give $28/3$. Finally the area of region A is 6 and of region B is 2, so the total area is 8. Therefore the center of mass is at $(7/6, -5/24)$.

An alternate way to do this is to note that the center of mass of region A is $(2, -1/2)$ and of region B is $(-4/3, 2/3)$. Then take the weighted average of these two points, where the weights are the areas of A and B.

3. First, divide the denominator into the numerator to get

$$4x + 4 + \frac{12x - 4}{4x^2 - 4x + 1}$$

The denominator is $(2x-1)^2$, so the fraction (by partial fractions) has the form $A/(2x-1)^2 + B/(2x-1)$. This decomposition is

$$\frac{2}{(2x-1)^2} + \frac{6}{2x-1}$$

This gives four routine integrals, giving $2x^2 + 4x - 1/(2x-1) + 3\ln(2x-1)$.

4. (a) One application of L'Hospital gives $-\sqrt{3}/2$.

(b) Write

$$\lim_{x\to\infty} x \tan\frac{1}{x} = \lim_{t\to 0} \frac{\tan t}{t} = 1.$$

(c) Write

$$\frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\frac{9+1/x}{1+1/x}}$$

which clearly goes to 3 as x goes to ∞.

5. (a) Powers beat polynomials, so this limit is 0.

   (b) The first factor is $1/2 + 1/(2n)$ which goes to $1/2$. The second clearly goes to 1, so the limit is $1/2$.

   (c) This is the limit of $\ln \frac{n}{n+1}$, which tends to $\ln 1 = 0$.

6. Taking the ratio of coefficients C_n to C_{n+1} (and canceling the factorials as needed) gives

$$\frac{(n+1)(n+2)}{(2n+2)(2n+1)}$$

which tends to $1/4$. This will be the radius of convergence.

Now consider the series

$$\sum_{n=0}^{\infty} \frac{C_n}{4^n}$$

Using the asymptotic expression for $C_n/4^n$ given in the Stirling's Approximation handout, the terms of this series are asymptotically equivalent to

$$\frac{1}{(n+1)\sqrt{n\pi}}$$

This series will then converge (absolutely) by comparison with the p-series $\sum n^{-3/2}$.

7. Using the generalized binomial theorem, the degree 3 polynomial for $(1-x)^{-1/2}$ is

$$1 + (1/2)x + (3/8)x^2 + (15/48)x^3$$

and for $\cos x$ is

$$1 - (1/2)x^2$$

Multiplying these and taking the degree 3 polynomial gives

$$1 + (1/2)x - (1/8)x^2 + (1/16)x^3$$

8. (a) Using the integral test, this series converges (absolutely).

   (b) Compare to the series $\sum(1/(n-1)!)$, which converges (absolutely).

   (c) By comparing the terms to $n^{-1/3}$, which tend to 0, the series will converge by the alternating series test. But also compare with the p-series $\sum(1/n^{1/3})$, which diverges. Therefore convergence is conditional.

9. (a) Computing a_{n+1}/a_n and taking the limit gives a radius of 1. Since $1/\sqrt{n}$ tends to 0, the series converges at the left endpoint by the alternating series test. It diverges at the right endpoint (p series, with $p = -1/2$).

   (b) Computing a_{n+1}/a_n and taking the limit gives a radius of 1. Since a_n tends to ∞, the series diverges at both endpoints.

10. Write this as

$$\sum_{n=1}^{\infty} \frac{n}{n+1} y^n$$

where $y = x/(2x+1)$. This power series in y will have a radius of convergence of 1, since $a_n/a_{n+1} \to 1$. It therefore remains to solve the inequality

$$\left|\frac{x}{2x+1}\right| < 1$$

and determine convergence when equality occurs. There are 4 values of x of interest. These are $x = 0$ (the numerator is 0), $x = -1/3$ (the value of the fraction is -1), $x = -1/2$ (the denominator is 0), and $x = -1$ (the value of the fraction is $+1$). These four values split the real line into five regions:

$$\begin{aligned}
\left|\frac{x}{2x+1}\right| &< 1 \qquad \text{if } x > 0 \\
\left|\frac{x}{2x+1}\right| &< 1 \qquad \text{if } -1/3 < x < 0 \\
\left|\frac{x}{2x+1}\right| &< 1 \qquad \text{if } x < -1 \\
\left|\frac{x}{2x+1}\right| &> 1 \qquad \text{if } -1 < x < -1/2 \\
\left|\frac{x}{2x+1}\right| &> 1 \qquad \text{if } -1/2 < x < -1/3
\end{aligned}$$

In the first three cases, the series converges absolutely; in the last two cases, it diverges. At $x = -1$, the series diverges, since the terms are $n/(n+1)$. At $x = -1/2$, the series is not defined. At $x = -1/3$, the series is alternating, but the terms are again $n/(n+1)$, so it diverges. And at $x = 0$, the series equals 0, and so converges absolutely.

11. (a) The ratio test gives the radius as $1/2$.

    (b) Define $F(x)$ to be this series. Then

$$\begin{aligned}
\int F(x) &= \int \sum_{n=0}^{\infty} (n+1)(-2x)^n \\
&= -\frac{1}{2}\sum_{n=0}^{\infty}(-2x)^{n+1} \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}(-2x)^{n} \\
&= \frac{x}{1+2x}
\end{aligned}$$

    Now differentiate to get

$$F(x) = \frac{1}{(1+2x)^2}$$

12. Set $t = 2 - y$ and substitute into the other equation to get $(x-2)(2-y) = 1$.

13. The polar equation is $r^2 \sin\theta\cos\theta = 1$, or

$$r^2 = 2\csc 2\theta$$

Now integrate $r^2/2$ between $\pi/6$ and $\pi/3$:

$$\int_{\pi/6}^{\pi/3} \csc 2\theta \, d\theta = -\frac{1}{2}\ln(\csc 2\theta + \cot 2\theta)\Big|_{\pi/6}^{\pi/3} = \ln\sqrt{3}$$

14. Computing the Cartesian forms for the derivatives gives $x'(t) = \sinh t$, $y'(t) = 1$, $x''(t) = \cosh t$, and $y''(t) = 0$. The speed is then $\sqrt{\sinh^2 t + 1} = \cosh t$. The tangential acceleration is the derivative of the speed, which is $\sinh t$. The normal component is $\sqrt{\cosh^2 t - \sinh^2 t} = 1$. The curvature is $-1/\cosh^2 t$.

15. Converting the line to parametric form gives $x = 3t + 2$, $y = 2t - 1$ and $z = t + 1$. The plane to be constructed will be parallel to the normal of the given plane, and also parallel to the direction vector of the line. Its normal will be perpendicular to these vectors, so a normal can be constructed by taking the cross product of $(3, 2, 1)$ with $(2, 0, 1)$, which is $(2, -1, -4)$. Since the point $(2, -1, 1)$ must be on this plane, inserting these into the form $2x - y - 4z$ gives $2x - y - 4z = 1$.

16. The obvious parametrization for this line is to set $t = x$. Then $x = t$, $y = -2 - t$ and $z = -1/3 - t/3$. Letting $Q = (0, -2, -1/3)$, $P = (-1, 0, 1)$ and $D = (1, -1, -1/3)$, we get $D \times (P - Q) = (-2/3, -1, 1)$. Therefore the distance between the point and the line is the length of this vector divided by the length of the direction vector, or $\sqrt{22/19}$.

17. (a) Replacing z with c, then putting the equation into ellipse form gives $a = \sqrt{1 - c^2/9}$ and $b = 2\sqrt{1 - c^2/9}$. Then the area is $\pi 2(1 - c^2/9)$.

    (b) Integrate $\pi 2(1 - z^2/9)\, dz$ from -3 to 3 to get 8π. (If is done with a general ellipsoid, one gets $(4/3)\pi abc$, where a, b and c are the semiaxes. Note that if $a = b = c = r$, one gets the formula for the volume of a sphere.)

18. Set $x = 2\sqrt{t}\cos t$, $y = 3\sqrt{t}\sin t$ and $z = \sqrt{1 - t}$. Then note that $x^2/4 + y^2/9 = t$ and $z^2 = 1 - t$. Therefore

$$x^2/4 + y^2/9 + z^2 = 1$$

and the curve lies on an ellipsoid.