Math 1572H
Stirling's approximation to $n!$

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April 25, 2008

Asymptotic analysis in probability and combinatorics requires evaluation of $n! = n(n-1)\ldots 2 \cdot 1$ for large n. This quantity is cumbersome to work with, but fortunately there is a remarkable approximation due to James Stirling.

To see that such an approximation might be found, consider approximating the log of $n!$ with an integral. That is,

$$\ln n! = \sum_{k=1}^{n} \ln k\,.$$

But this is a lower sum for

$$\int_1^{n+1} \ln x\,dx = (n+1)\ln(n+1) - n\,.$$

Exponentiating gives

$$n! < \frac{(n+1)^{n+1}}{e^n}\,.$$

This approximation is not very good, but it gives us an idea of how to approach $n!$.

Instead of this crude lower sum, let's take a more refined upper sum for the integral

$$\int_1^{n} \ln x\,dx\,.$$

We break the interval $[1, n]$ into subintervals: $[1, 3/2]$, $[3/2, 5/2]$, $[5/2, 7/2]$, $\ldots$, $[n-1/2, n]$. Each subinterval has width 1 except the first and last. For each of the width 1 subintervals, form the trapezoid whose top is tangent to the graph of $\ln x$ at k, for each $k = 2, 3, \ldots, (n-1)$. Each of these trapezoids will have area $\ln k$. For the first subinterval (width 1/2), form a rectangle of height 2. This will have area 1. For the last subinterval, form a rectangle of height $\ln n$. This will have area

$(\ln n)/2$. Summing and recognizing this as an upper sum, we have

$$\begin{aligned}\int_1^n \ln x\,dx &= n\ln n - n + 1\\ &= \ln((n/e)^n) + 1\\ &< 1 + \ln 2 + \ln 3 + \cdots + \ln(n-1) + (\ln n)/2\\ &= \ln n! - (\ln n)/2 + 1\,.\end{aligned}$$

Passing to exponentials, we have

$$x_n = \frac{(n/e)^n\sqrt{n}}{n!} < 1\,.$$

Thus the sequence $\{x_n\}$ is bounded above by 1. We next show that this sequence is monotone increasing. Consider the ratio x_{n+1}/x_n.

$$\frac{x_{n+1}}{x_n} = \frac{\frac{\left(\frac{n+1}{e}\right)^{n+1}\sqrt{n+1}}{(n+1)!}}{\frac{\left(\frac{n}{e}\right)^n\sqrt{n}}{n!}} = \frac{(1+1/n)^{n+1/2}}{e}\,.$$

To show $\{x_n\}$ is increasing, we must show $(1+1/n)^{n+1/2} > e$. Consider the integral of $1/x$ between n and $n+1$. Form the trapezoid whose top is tangent to $1/x$ at $n+1/2$. The area of this trapezoid is $1/(n+1/2) = 2/(2n+1)$. Since this trapezoid lies below the curve $1/x$, we have

$$\int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n > \frac{2}{2n+1}\,,$$

or

$$\ln(1+1/n) > \frac{2}{2n+1}\,.$$

Putting the fraction on the other side of the inequality gives

$$(n+1/2)\ln(1+1/n) > 1\,.$$

Now pass to exponentials to get

$$(1+1/n)^{n+1/2} > e\,,$$

as required.

Therefore we have that the sequence $\{x_n\}$ is monotone increasing and bounded above by 1. Therefore it approaches some limit $L \leq 1$. Our final task is to evaluate L.

To this end, we will need Wallis's formula for π. Please read pages 380–382 in the textbook, which gives a careful proof of this formula:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots\,.$$

But then

$$\frac{\pi}{2} = \lim_{n\to\infty}\frac{2^2}{3^2}\cdot\frac{4^2}{5^2}\cdot\frac{6^2}{7^2}\cdots\frac{(2n-2)^2}{(2n-1)^2}\cdot 2n\,.$$

Now take square roots to get

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-1}\cdot\sqrt{2n}\,.$$

Multiply numerator and denominator by $2\cdot 4\cdot 6\ldots(2n-2)$ to get

$$\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-1} = \frac{((n-1)!)^2 2^{2(n-1)}}{(2n-1)!}\,.$$

Then

$$\begin{aligned}\sqrt{\pi} &= \lim_{n\to\infty} 2\sqrt{n}\cdot\frac{((n-1)!)^2 2^{2(n-1)}}{(2n-1)!}\\ &= \lim_{n\to\infty}\sqrt{n}\cdot\frac{(n!)^2 2^{2n}}{(2n-1)!2n^2}\\ &= \lim_{n\to\infty}\frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}}\,.\end{aligned}$$

Since x_n tends to L, so will x_{2n}. For n sufficiently large, both x_n and x_{2n} will be near L, so the ratio x_n/x_{2n} will be near 1. Therefore

$$\lim_{n\to\infty}\frac{x_n^2}{x_{2n}} = L\,.$$

But we have

$$\begin{aligned}\lim_{n\to\infty}\frac{x_n^2}{x_{2n}} &= \lim_{n\to\infty}\frac{(n/e)^{2n} n(2n)!}{(2n/e)^{2n}\sqrt{2n}(n!)^2}\\ &= \lim_{n\to\infty}\frac{(2n)!\sqrt{n}}{\sqrt{2}(n!)^2 2^{2n}}\\ &= \frac{1}{\sqrt{2\pi}}\,.\end{aligned}$$

Putting this together gives

Theorem 1.

$$\lim_{n\to\infty}\frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1\,.$$

In other words, $n^n e^{-n}\sqrt{2\pi n}$ is an approximation to $n!$ for large n.

As an example, let's estimate the likelihood that a random sequence of $2n$ parentheses is "well-formed," that is, parentheses sequences that can be paired and nested. We need the following fact about well-formed parentheses. The number of such sequences is the Catalan number, $\binom{2n}{n}/(n+1)$. There are 2^{2n} possible sequences of $2n$ parentheses, and assuming each is equally likely, the probability that a sequence will be well-formed is

$$P_n = \frac{\binom{2n}{n}}{(n+1)2^{2n}}\,.$$

Now replace the binomial coefficient with $(2n)!/(n!n!)$ and use Stirling's approximation for all the factorials to get

$$\begin{aligned}P_n &\sim \frac{(2n)^{2n}e^{-2n}2\sqrt{n\pi}}{(n+1)n^{2n}e^{-2n}2n\pi 2^{2n}} \\ &= \frac{1}{(n+1)\sqrt{n\pi}}\,.\end{aligned}$$