A Generalization of Suter's Surprising Cyclic Symmetry and an Associated CSP

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Outline

Suter's Symmetry: Goal

Goal

Define a symmetry on the subposet of Young's lattice consisting of those partitions with hook lengths less than or equal to k.

Partitions

Definition

A **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a finite sequence of weakly decreasing positive integers.

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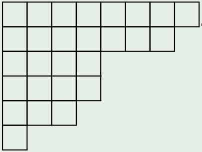
A **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a finite sequence of weakly decreasing positive integers.

Partitions have an associated Ferrers diagram of boxes, where the number of boxes in the ith row is equal to the ith integer in the sequence.

Example

The partition $(8, 7, 4, 4, 3, 1)$ is represented as .

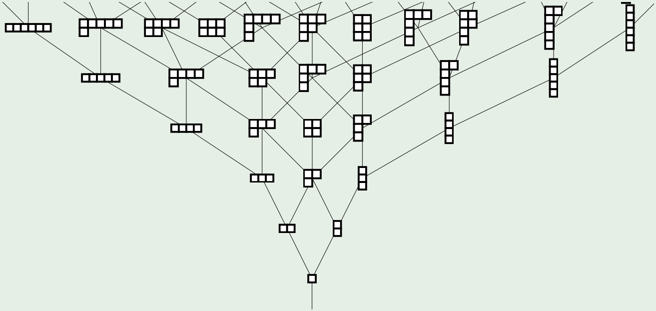
Young's Lattice

Definition

Young's lattice is the poset containing all partitions ordered by inclusion of their Ferrers diagrams.

Example

The first few ranks of the Hasse diagram for Young's lattice are

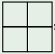
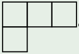


R. Suter's Poset

Definition (R. Suter)

Let $\mathcal{Y}_2^k$ be the subposet of Young's lattice containing those partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell)$ for which $\lambda_1 + \ell \leq k + 1$.

Example

$\mathcal{Y}_2^3$ contains , , and —but not .

R. Suter's Poset

Theorem (R. Suter)

The graph of $\mathcal{Y}_2^k$ has $(k+1)$-fold cyclic symmetry.

R. Suter's Poset

Theorem (R. Suter)

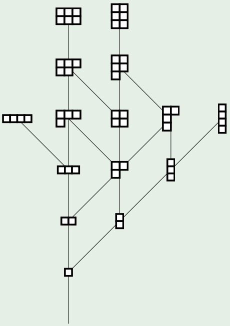
The graph of $\mathcal{Y}_2^k$ has $(k+1)$-fold cyclic symmetry.

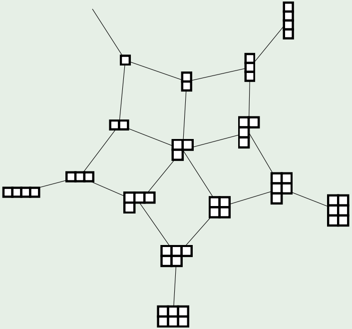
Proof idea: R. Suter gives an explicit cyclic action on partitions to realize this symmetry.

R. Suter's Symmetry

We see the symmetry by drawing the Hasse diagram of $\mathcal{Y}_2^k$ and then remembering only the underlying graph structure. The picture below illustrates this for $k = 4$.

Example





Motivation

Motivation

Is there an easy way to understand this symmetry?

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Is there an easy way to understand this symmetry?

Yes, but first we will generalize.

Berg and Zabrocki's Symmetry: Goal

Goal

Generalize the previous symmetry to the subposet of k-Young's lattice consisting of those $(k+1)$-cores contained in a product of rectangles.

Cores

Definition

The **hook length** of a box in a Ferrers diagram is the number of boxes k below and to the right of that box.

Example

The hook lengths of $(5, 3, 1, 1)$ are given by

$$\begin{array}{|c|c|c|c|c|}\hline 8 & 5 & 4 & 2 & 1 \\ \hline 5 & 2 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

Cores

Definition

The **hook length** of a box in a Ferrers diagram is the number of boxes k below and to the right of that box.

Definition

A $(k+1)$-**core** is a partition with no hook length of size $(k+1)$.

Example

$(5,3,1,1)$ is a 3-core:

8	5	4	2	1
5	2	1		
2				
1				

C. Berg and M. Zabrocki's Poset

Definition (C. Berg, M. Zabrocki)

For fixed k, let $R_i = (i^{k-i+1})$ be the rectangular partition with $k - i + 1$ parts of size i.

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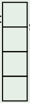
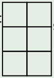
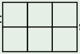
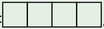
Definition (C. Berg, M. Zabrocki)

For fixed k, let $R_i = (i^{k-i+1})$ be the rectangular partition with $k - i + 1$ parts of size i.

Example

For $k = 4$, we have the partitions

$R_1 =$ , $R_2 =$ , $R_3 =$ , and $R_4 =$ .

C. Berg and M. Zabrocki's Poset

Definition (C. Berg, M. Zabrocki)

For fixed k, m and for $i_1 \geq i_2 \geq \cdots \geq i_{m-1}$, let $R_{k,\{i_1,i_2,\ldots,i_{m-1}\}}$ be the Ferrers diagram obtained by placing the Ferrers diagram of $R_{k,i_{j+1}}$ at the lower left corner of the Ferrers diagram of R_{k,i_j}.

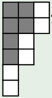
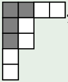
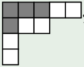
C. Berg and M. Zabrocki's Poset

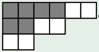
Example

For $k = 2$ and $m = 4$, we have

$R_1 =$ , $R_2 =$ ,

so that

$R_{2,\{1,1,1\}} =$ , $R_{2,\{2,1,1\}} =$ , $R_{2,\{2,2,1\}} =$ ,

$R_{2,\{2,2,2\}} =$ .

C. Berg and M. Zabrocki's Poset

Definition (C. Berg, M. Zabrocki)

Let $\mathcal{Y}_m^k$ be the subposet of the Young's lattice consisting of all $(k+1)$-cores contained in some $R_{k,\{i_1,i_2,\ldots,i_{m-1}\}}$.

C. Berg and M. Zabrocki's Poset

Definition (C. Berg, M. Zabrocki)

Let $\mathcal{Y}_m^k$ be the subposet of the Young's lattice consisting of all $(k+1)$-cores contained in some $R_{k,\{i_1,i_2,\ldots,i_{m-1}\}}$.

When $m = 2$, we are limited to a single rectangle (i^{k-i+1}). In that case, the definition coincides with R. Suter's condition that $\lambda_1 + \ell \leq i + (k - i + 1) = k + 1$.

C. Berg and M. Zabrocki's Poset

Theorem (C. Berg, M. Zabrocki)

The graph of $\mathcal{Y}_m^k$ *has* $(k+1)$*-fold cyclic symmetry.*

C. Berg and M. Zabrocki's Poset

Theorem (C. Berg, M. Zabrocki)

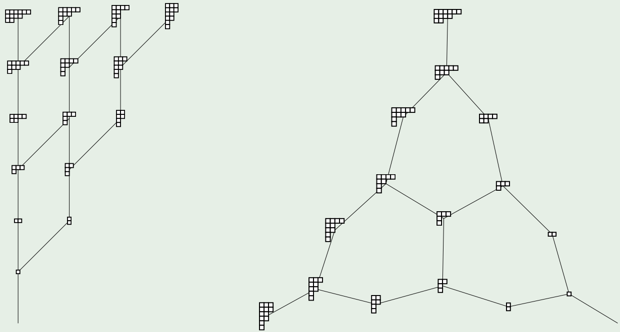
The graph of $\mathcal{Y}_m^k$ has $(k+1)$-fold cyclic symmetry.

Proof idea: C. Berg and M. Zabrocki use geometry.

C. Berg and M. Zabrocki's Symmetry

We see the symmetry by drawing the Hasse diagram of $\mathcal{Y}_m^k$ and then remembering only the underlying graph structure. The picture below illustrates this for $k = 2$ and $m = 4$.

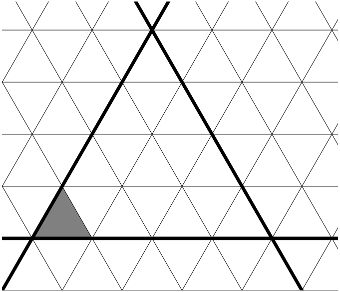
Example



Geometric Proof

Definition

In $\mathbb{R}^{k+1}$, the type A **affine hyperplane arrangement** is the set of hyperplanes $\{x_i - x_j = p : 1 \leq i < j \leq k+1 \text{ and } p \in \mathbb{Z}\}$. The **dominant cone** is the collection of points such that $x_1 > x_2 > \cdots > x_{k+1}$.



Geometric Proof

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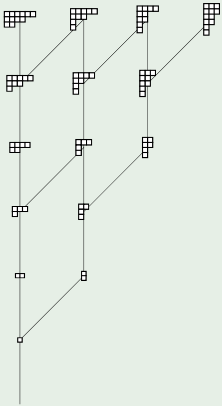
Proposition

There is a bijection between $\mathcal{Y}_m^k$ and alcoves in the dominant cone bounded by the hyperplane $x_1 - x_{k+1} = m$.

Geometric Example

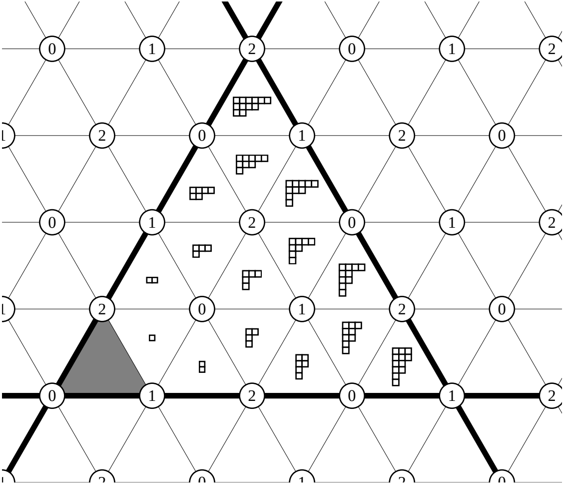
Example

The poset $\mathcal{Y}_4^2$:

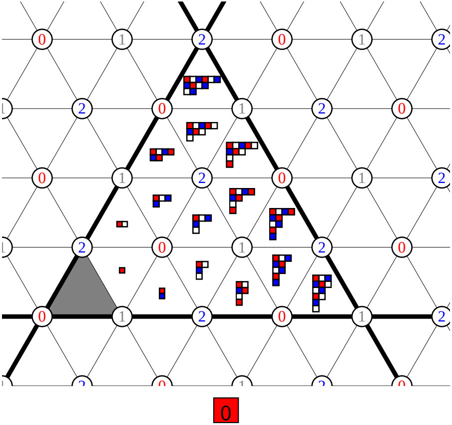


Geometric Example

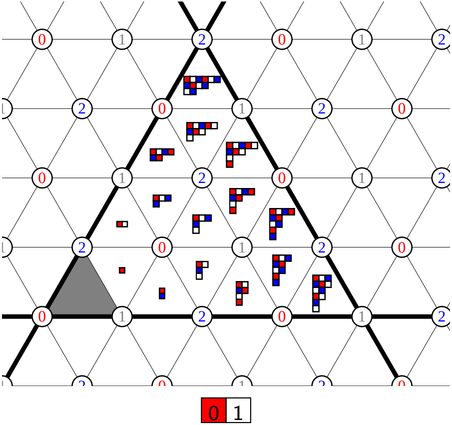
We may restrict to the plane $x_1 + x_2 + x_3 = 0$ to draw in $\mathbb{R}^2$:



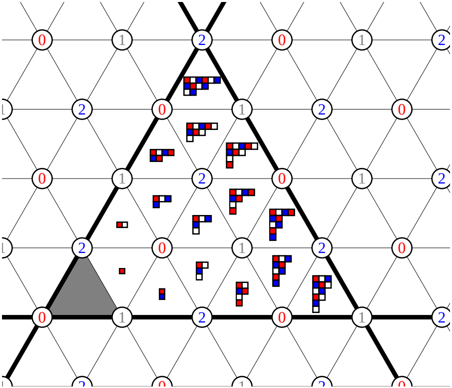
Geometric Example

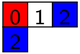


Geometric Example

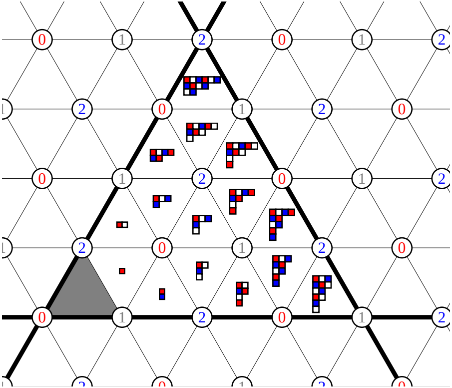


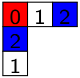
Geometric Example





Geometric Example





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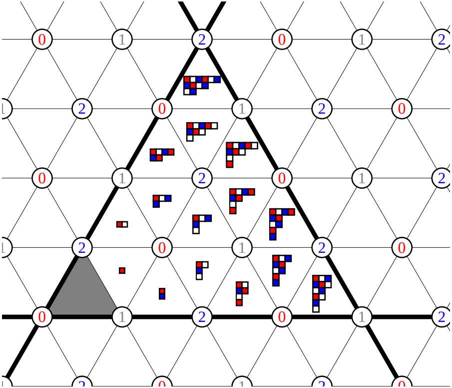
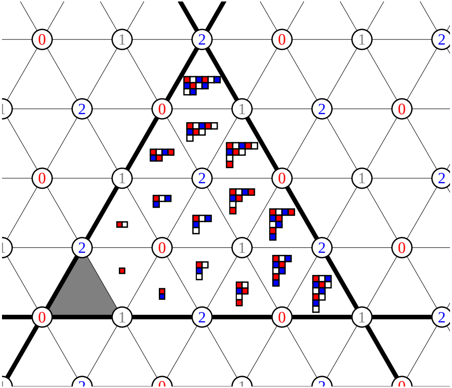
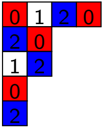


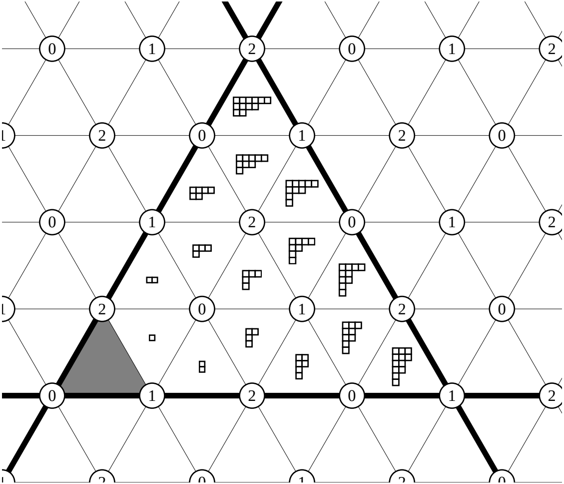
0	1	2	0
2	0		
1			
0			

Geometric Example





Geometric Example



Motivation

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Is there an easy way to understand this symmetry?

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Is there an easy way to understand this symmetry?

Yes, but first we must combinatorialize.

A Combinatorial Model: Goals

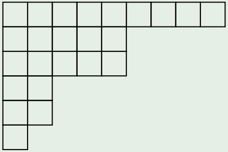
Goals

1. Reduce the definitions of the poset $\mathcal{Y}_m^k$ to simple rules for words on $\{0, 1, \ldots, m-1\}$ of length k.
2. Give an explicit description of the symmetry of this poset.

Cores to Words

Form the Ferrers diagram of a **core** $\lambda \in \mathcal{Y}_m^k$.

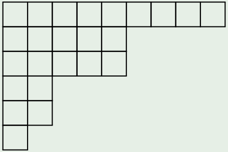
Example

(1) Consider the 5-**core**  in $\mathcal{Y}_4^4$.

Cores to Words

Encode this diagram as a **boundary word** by tracing along the diagram from the top right to the bottom left, recording 1 for each step left and 0 for each step down.

Example

(1) Consider the 5-**core**  in $\mathcal{Y}_4^4$.

(2) It has the **boundary word** $\ldots 011110011100101 \ldots$.

Cores to Words

Form an **abacus display** with $(k+1)$ runners (labeled $0, 1, \ldots, k$) by breaking the infinite binary word into consecutive runs of length $(k+1)$ and stacking them.

Example

(2) It has the **boundary word** $\ldots 011110011100101 \ldots$.

(3) We split this to make $\ldots|01111|00111|00101|\ldots$.

The **abacus display** is

0	1	1	1	1
0	0	1	1	1
0	0	1	0	1

Cores to Words

The 1s in the display of $(k+1)$-cores have columns **flush to the top**. We may just record the number of 1s in each column.

Example

(3) We split this to make $\ldots|01111|00111|00101|\ldots$.

(4) The corresponding **word** is

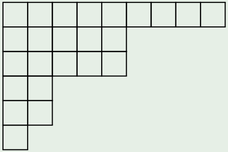
$$\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 3 & 2 & 3 \end{array}$$

Cores to Words

The **core** was contained in $R_{k,\{i_1,i_2,\ldots,i_{m-1}\}}$, so the display has at most $m-1$ rows and the **word** is on $\{0,1,\ldots,m-1\}$.

Example

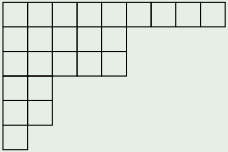
(4) The corresponding **word** is
$$\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 3 & 2 & 3 \end{array}$$

(5) The **core**  in $\mathcal{Y}_4^4$ maps to the **word** 1323.

Cores to Words

This is a bijection from $\mathcal{Y}_m^k$ to **words** of length k on $\{0, 1, \ldots, m-1\}$.

Example

(5) The **core**  in $\mathcal{Y}_4^4$ maps to the **word** 1323.

A Combinatorial Model

We can describe the poset $\mathcal{Y}_m^k$ purely in terms of these words.

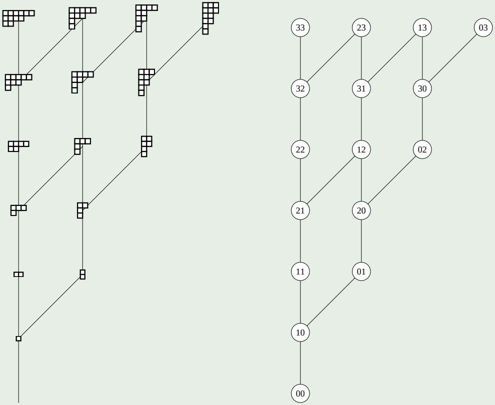
Definition (W.)

Let $\mathcal{X}_m^k$ be the poset of words $\mathbf{x}$ of length k on $[m]$ with the partial order induced by the following covering relations:

1. For $a < m-1$, $\mathbf{y}a \lessdot (a+1)\mathbf{y}$.
2. For $b < a$, $\mathbf{y}ab\mathbf{z} \lessdot \mathbf{y}ba\mathbf{z}$.

Example

The posets $\mathcal{X}_4^2$ and $\mathcal{Y}_4^2$:



Symmetry

Theorem (W.)

The graph of $\mathcal{X}_m^k$ has $(k+1)$-fold cyclic symmetry.

Symmetry

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The graph of $\mathcal{X}_m^k$ has $(k+1)$-fold cyclic symmetry.

Proof idea: we define a cyclic action of order $(k+1)$ that is a graph isomorphism.

Symmetry

Definition

1. Given a word $\mathbf{x} \in \mathcal{X}_m^k$, form the extended word

$$\overline{\mathbf{x}} = (\mathbf{x})(m-1)(\mathbf{x}-1)(m-2)\ldots(\mathbf{x}-m+1)(0).$$

Symmetry

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2. Act on $\bar{\mathbf{x}}$ by cyclically rotating it left so that its leftmost 0 appears as its rightmost character.

Symmetry

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$$\bar{\mathbf{x}} = (\mathbf{x})(m-1)(\mathbf{x}-1)(m-2)\ldots(\mathbf{x}-m+1)(0).$$

2. Act on $\bar{\mathbf{x}}$ by cyclically rotating it left so that its leftmost 0 appears as its rightmost character.
3. This induces an action on $\mathcal{X}_m^k$ by restricting the resulting word to its first k letters.

Symmetry

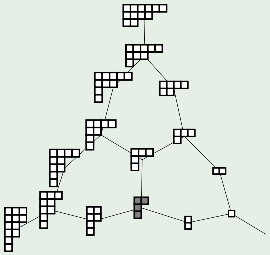
Proof.

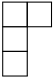
1. This is a cyclic action of order $(k+1)$, since there are $(k+1)$ zeros in $\bar{\mathbf{x}}$.
2. It is easy to show that this action takes edges to edges.

□

The Boundary Path for $m = 4, k = 2$

Example

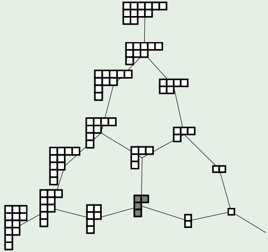


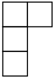


Map from the core to the word.

The Boundary Path for $m = 4, k = 2$

Example

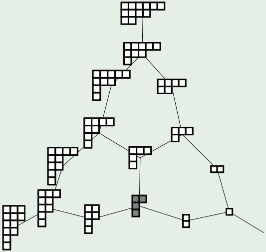


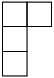
 $\rightarrow 010|010|000$

Map from the core to the word.

The Boundary Path for $m = 4, k = 2$

Example

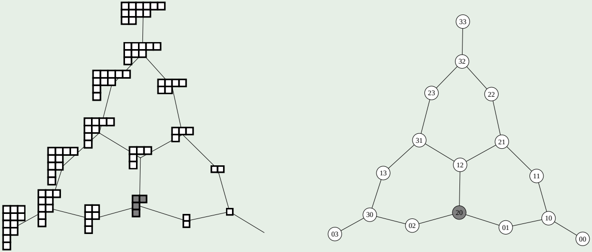


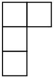
 $\rightarrow 010|010|000 \rightarrow \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$

Map from the core to the word.

The Boundary Path for $m = 4, k = 2$

Example



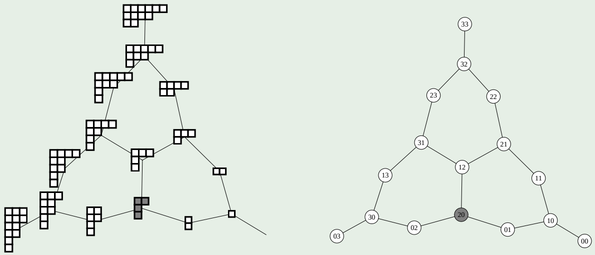


$$\rightarrow 010|010|000 \rightarrow \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \rightarrow 20$$

Map from the core to the word.

The Boundary Path for $m = 4, k = 2$

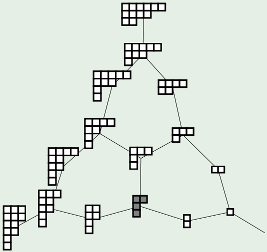
Example

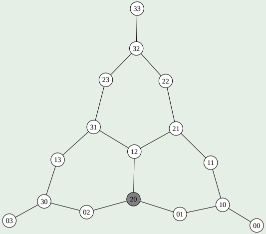


20

The Boundary Path for $m = 4, k = 2$

Example





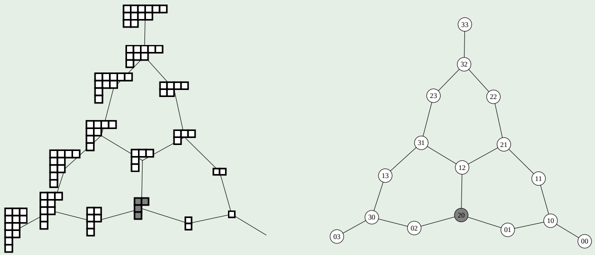
203 132 021 310

Form the extended word

$$\bar{\mathbf{x}} = (\mathbf{x})(m-1)(\mathbf{x}-1)(m-2)\ldots(\mathbf{x}-m+1)(0).$$

The Boundary Path for $m = 4, k = 2$

Example

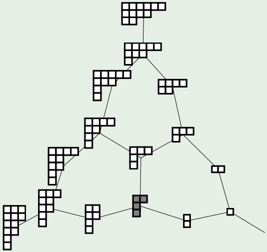


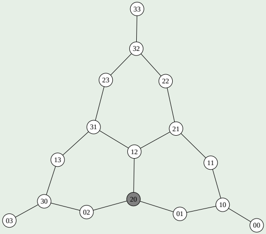
$\underline{20}3\ 132\ 021\ 310$

Act on $\overline{\mathbf{x}}$ by cyclically rotating it left so that its leftmost 0 appears as its rightmost character.

The Boundary Path for $m = 4, k = 2$

Example



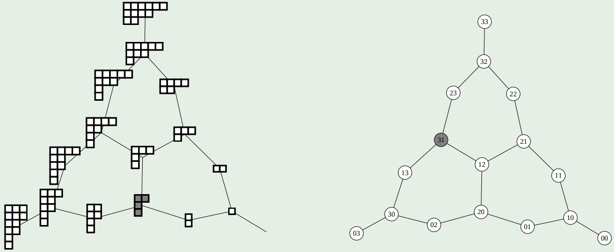


$313\ 202\ 131\ 0\underline{20}$

Act on $\bar{\mathbf{x}}$ by cyclically rotating it left so that its leftmost 0 appears as its rightmost character.

The Boundary Path for $m = 4, k = 2$

Example

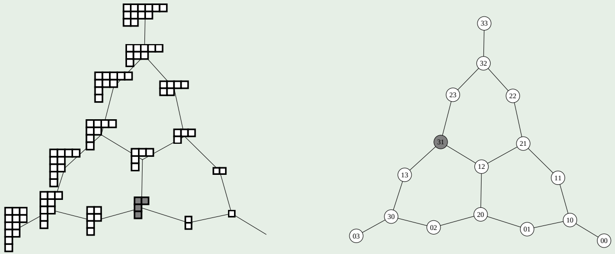


313 202 131 020

This induces an action on $\mathcal{X}_m^k$ by restricting the resulting word to its first k letters.

The Boundary Path for $m = 4, k = 2$

Example

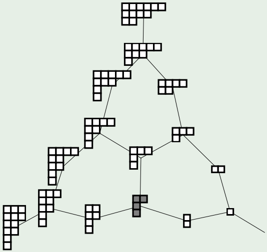


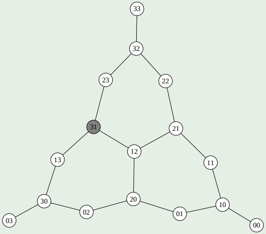
$$031 \to \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}$$

Map from the new word back to a core.

The Boundary Path for $m = 4, k = 2$

Example



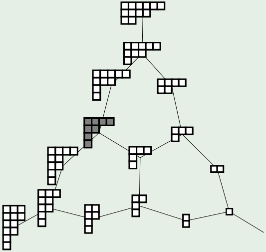


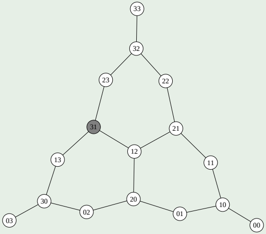
$$031 \to \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{matrix} \to 011|010|010$$

Map from the new word back to a core.

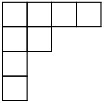
The Boundary Path for $m = 4, k = 2$

Example





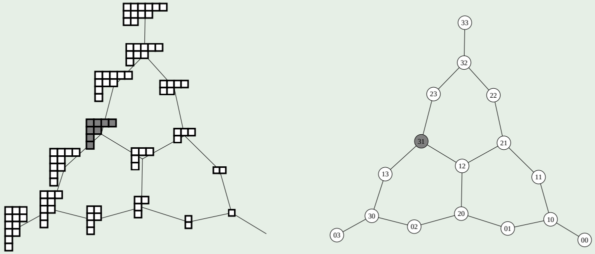
$$031 \to \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{matrix} \to 011|010|010 \to$$



Map from the new word back to a core.

The Boundary Path for $m = 4, k = 2$

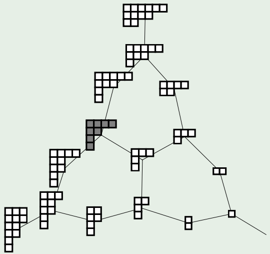
Example

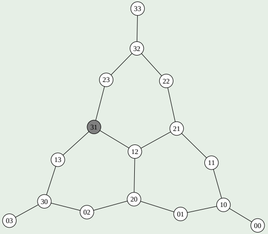


31

The Boundary Path for $m = 4, k = 2$

Example





313 202 131 020

Form the extended word

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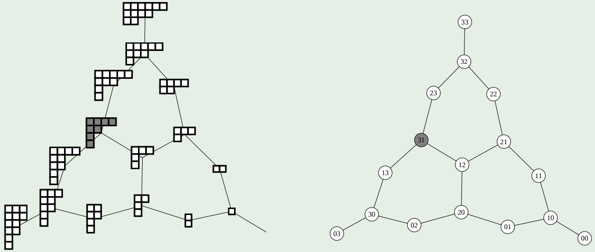
The Boundary Path for $m = 4, k = 2$

Example

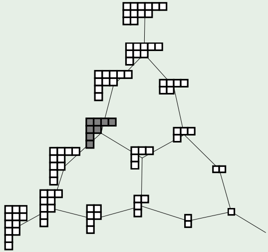


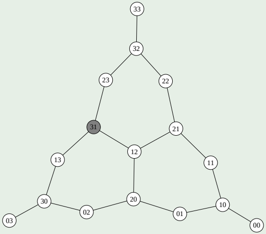
313 202 131 020

Act on $\bar{\mathbf{x}}$ by cyclically rotating it left so that its leftmost 0 appears as its rightmost character.

The Boundary Path for $m = 4, k = 2$

Example



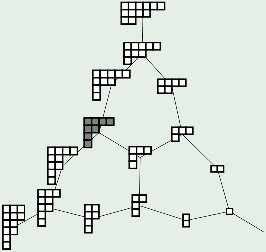


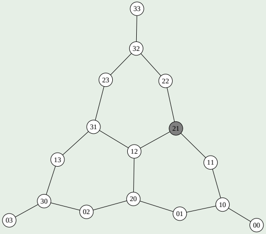
213 102 031 320

Act on $\bar{\mathbf{x}}$ by cyclically rotating it left so that its leftmost 0 appears as its rightmost character.

The Boundary Path for $m = 4, k = 2$

Example



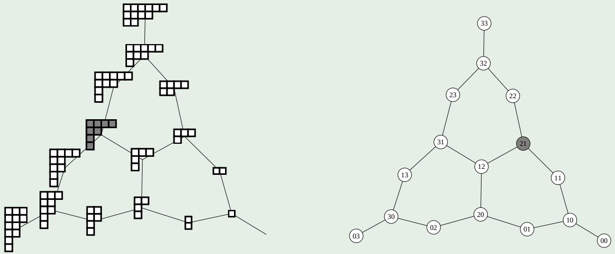


213 102 031 320

This induces an action on $\mathcal{X}_m^k$ by restricting the resulting word to its first k letters.

The Boundary Path for $m = 4, k = 2$

Example

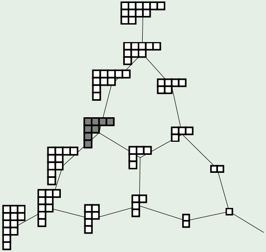


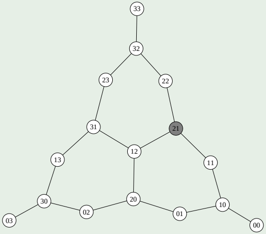
$$021 \to \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Map from the new word back to a core.

The Boundary Path for $m = 4, k = 2$

Example



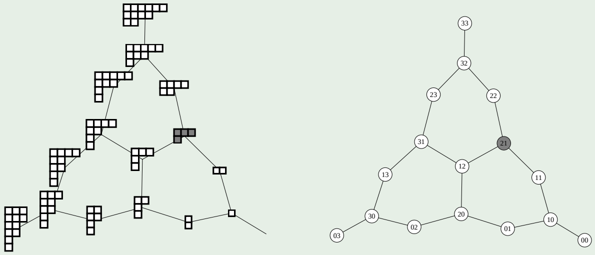


$$021 \to \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \to 011|010|000$$

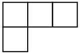
Map from the new word back to a core.

The Boundary Path for $m = 4, k = 2$

Example



$$021 \to \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \to 011|010|000 \to$$



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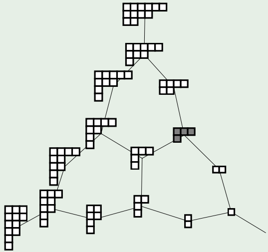

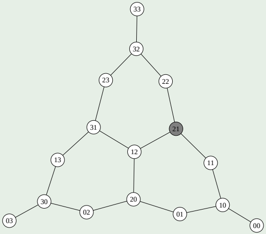

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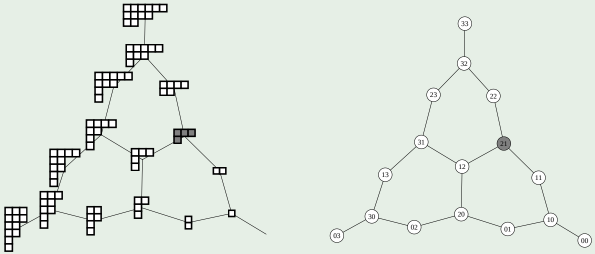

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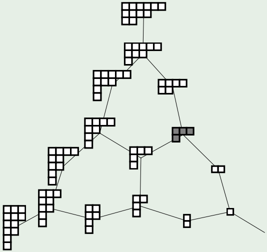
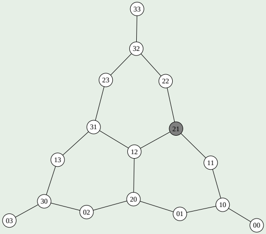
The Boundary Path for $m = 4, k = 2$

Example



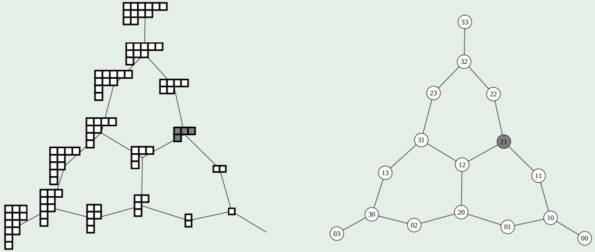


213 102 031 320

Act on $\bar{\mathbf{x}}$ by cyclically rotating it left so that its leftmost 0 appears as its rightmost character.

The Boundary Path for $m = 4, k = 2$

Example



203 132 021 310

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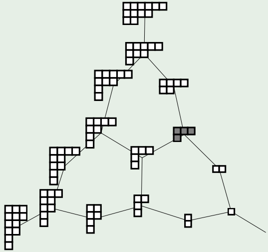
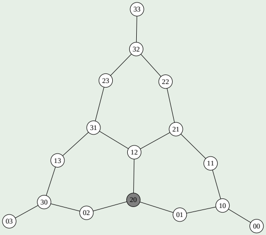
Example





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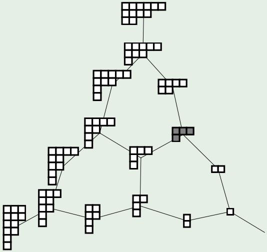
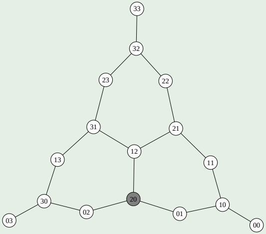


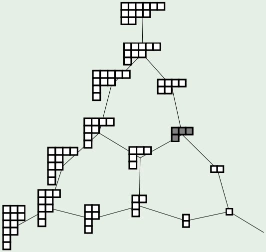


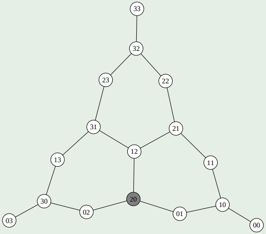
$$
\begin{array}{cccc}
 & 0 & 1 & 0 \\
20 \rightarrow & 0 & 1 & 0 \\
 & 0 & 0 & 0
\end{array}
$$

Map from the new word back to a core.

The Boundary Path for $m = 4, k = 2$

Example



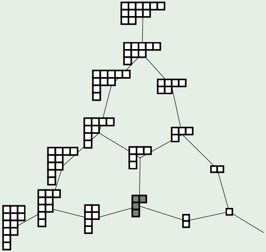


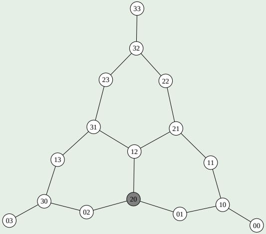
$$\begin{matrix} & & 0 & 1 & 0 & & \\ 20 & \to & 0 & 1 & 0 & \to & 010|010|000 \\ & & 0 & 0 & 0 & & \end{matrix}$$

Map from the new word back to a core.

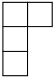
The Boundary Path for $m = 4, k = 2$

Example





$$20 \to \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \to 010|010|000 \to$$



Map from the new word back to a core.

Motivation

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Is there an easy way to understand this symmetry?

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2. We will define an equivariant map from $\mathcal{X}_m^k$ to words on $\{0, 1, \ldots, m-1\}$ of length $(k+1)$ that sum to $m-1 \pmod{m}$ under rotation.

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1. Yes.
2. We will define an equivariant map from $\mathcal{X}_m^k$ to words on $\{0, 1, \ldots, m-1\}$ of length $(k+1)$ that sum to $m-1 \pmod{m}$ under rotation.
3. The orbit structures of these words can be understood using the cyclic sieving phenomenon.

Bijactions: Goal

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Define the forward direction of an equivariant bijection from $\mathcal{X}_m^k$ to words on $\{0, 1, \ldots, m-1\}$ of length $(k+1)$ that sum to $m-1 \pmod{m}$ under rotation.

Bijactions: Goal

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Define the forward direction of an equivariant bijection from $\mathcal{X}_m^k$ to words on $\{0, 1, \ldots, m-1\}$ of length $(k+1)$ that sum to $m-1 \pmod{m}$ under rotation.

The map will be defined in such a way that it will be obvious that it is equivariant, but not obvious that it is a bijection.

Bijactions

Let C_k be the cyclic group of order k and let X be a set acted on by C_k, with generator c.

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Definition (W.)

Let S be a set, let $s : X \to S$ be a statistic. Map $x \in X$ to the word $w(x)$ by

$$w(x) = s(x)s(c(x))s(c^2(x)) \ldots s(c^{k-1}(x)),$$

and take $W = w(X)$ to be the set of all such words.

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Take-away message

Fix an element. For each object in the orbit of that element, compute the chosen statistic. Concatenate these statistics into a word.

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1. By construction, w is an equivariant map to words under left rotation.
2. When w is injective, we call the equivariant bijection $w : X \to W$ a **bijaction**.

A Bijaction

Definition

Let $\mathcal{W}_m^k$ be the set of all words on $[m]$ of length $(k+1)$ with sum equal to $(m-1) \pmod{m}$.

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Forward map

Take the statistic to be the leftmost letter of $\mathbf{x} \in \mathcal{X}_m^k$.

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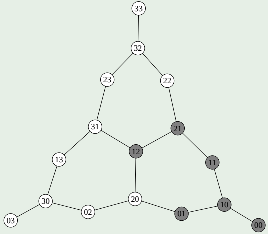
Backward map

IOU.

Example

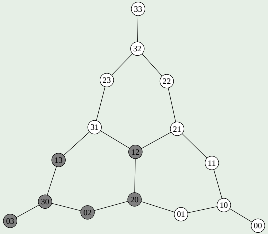
Example



00
10
01
11
21
12

Example

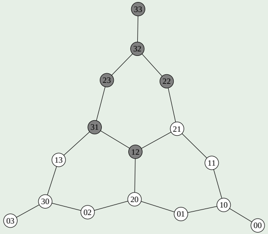
Example



0	**0**3
1	**3**0
0	**1**3
1	**0**2
2	**2**0
1	**1**2

Example

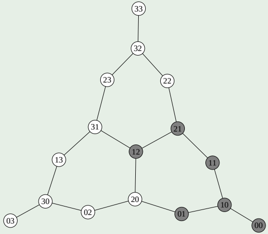
Example



00	**3**3
13	**3**2
01	22
10	23
22	**3**1
11	12

Example

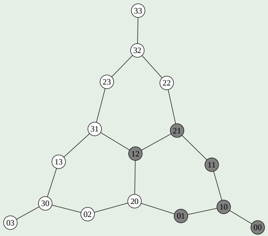
Example



003	00
133	10
012	01
102	11
223	21
111	12

Example

Example



$$\begin{array}{ccc}
\{\mathbf{00}, 03, 33\} & \xrightarrow{w} & \{\mathbf{003}, 030, 300\} \\
\{\mathbf{10}, 30, 32\} & \xrightarrow{w} & \{\mathbf{133}, 331, 313\} \\
\{\mathbf{01}, 13, 22\} & \xrightarrow{w} & \{\mathbf{012}, 120, 201\} \\
\{\mathbf{11}, 02, 23\} & \xrightarrow{w} & \{\mathbf{102}, 021, 210\} \\
\{\mathbf{21}, 20, 31\} & \xrightarrow{w} & \{\mathbf{223}, 232, 322\} \\
\{\mathbf{12}, 12, 12\} & \xrightarrow{w} & \{\mathbf{111}, 111, 111\}
\end{array}$$

Example

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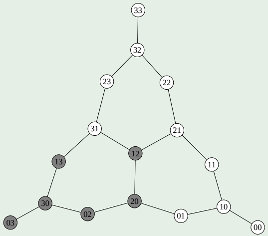

$$\begin{array}{ccc}
\{00, \mathbf{03}, 33\} & \xrightarrow{w} & \{003, \mathbf{030}, 300\} \\
\{10, \mathbf{30}, 32\} & \xrightarrow{w} & \{133, \mathbf{331}, 313\} \\
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\{11, \mathbf{02}, 23\} & \xrightarrow{w} & \{102, \mathbf{021}, 210\} \\
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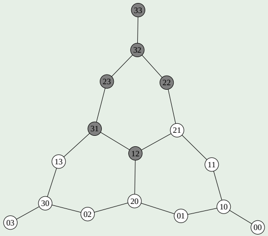

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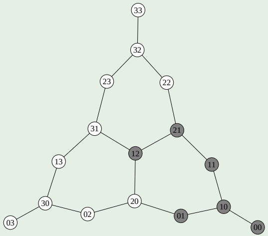

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1. We will first understand the orbit structure of $\mathcal{W}_m^k$ and
2. Then consider the inverse of this map.

The Cyclic Sieving Phenomenon: Goal

Goal

Apply the cyclic sieving phenomenon to understand the orbit structure $\mathcal{W}_m^k$ under rotation.

The Cyclic Sieving Phenomenon

Definition (V. Reiner, D. Stanton, D. White)

Let X be a finite set, $X(q)$ a generating function for X, and C_k a cyclic group acting on X. Then the triple $(X, X(q), C_k)$ exhibits the **Cyclic Sieving Phenomenon (CSP)** if for $c \in C_k$,

$$X(\omega(c)) = |\{x \in X : c(x) = x\}| ,$$

where $\omega : C_k \to \mathbb{C}$ is an isomorphism of C_k with the kth roots of unity.

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Take-away Message

We can sometimes understand the orbit structure of a set under a cyclic action by specializing a polynomial to roots of unity.

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We have five orbits of size 3:

$$\{003, 030, 300\}, \{133, 331, 313\}, \{012, 120, 201\},$$

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and one orbit of size 1:

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2. and $\mathcal{W}_4^2(\omega^3) = \mathcal{W}_4^2(1) = 16$ (all elements are fixed by three rotations).

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$(\mathcal{W}_m^k, \mathcal{W}_m^k(q), C_{k+1})$ exhibits the CSP.

Corollary (up to IOU)

$(\mathcal{Y}_m^k, \mathcal{W}_m^k(q), C_{k+1})$ exhibits the CSP.

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So is it?

Dendrodistinctivity: Goal

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2. The difficulty lies in showing that this additional information is not actually additional.
3. This is the form of the problem that I'll state.

Words with partitions

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3. We label these blocks from left to right by
$$\sigma - m + 1, \ldots, \sigma - 1, \sigma,$$
where $\sigma = \sum_{i=1}^{k} w_i \pmod{m}$.

Words with partitions

Example

For clarity, I'll use "$\cdot$" to indicate an empty word, and a "$|$" to indicate the end of a block. Here is an example for $m = 4$ and $k = 9$:

$$\mathbf{w} = 200310311,$$

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$$\mathbf{w} = 200310311,$$

$$\mathbf{w_0 w_1 w_2 w_3} = \underset{0}{2003}|\underset{1}{1}|\underset{2}{031}|\underset{3}{1}.$$

A Tree

Definition

Define an infinite complete m-ary tree $\underline{\mathcal{T}_m}$ on partitioned words $\underline{\mathbf{w}}$ by

1. The 0th level consists of the empty word $\cdot$, partitioned as $\underline{\cdot} = \underset{1}{\cdot}\big|\underset{2}{\cdot}\big|\cdots\big|\underset{0}{\cdot}$.
2. The children of a word $\underline{\mathbf{w}}$ are the m words obtained by prepending $-i$ to the ith block.

Example

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$\underline{\mathbf{w}} = \underset{0}{200}3|\underset{1}{1}|0\underset{2}{3}1|\underset{3}{1}$ is in $\underline{\mathcal{T}_4}$.

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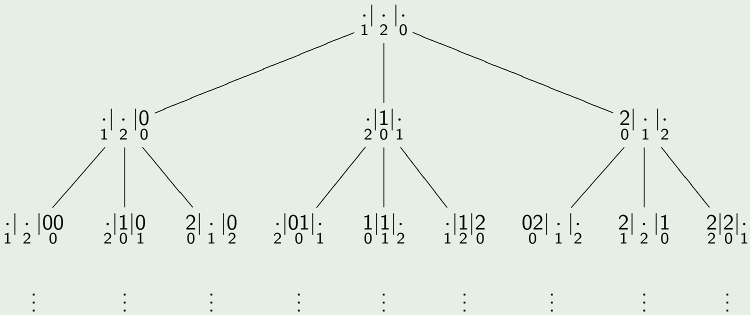
$$\underset{2}{2003}|\underset{3}{1}|\underset{0}{2031}|\underset{1}{1}$$

$$\underset{1}{2003}|\underset{2}{1}|\underset{3}{031}|\underset{0}{11}$$

An Example Tree

Example

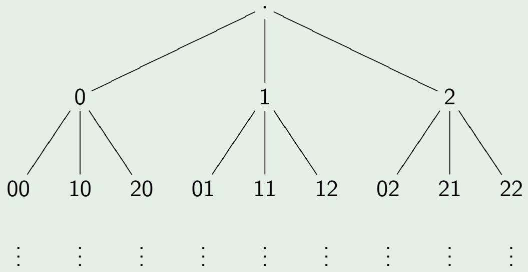
$\underline{\mathcal{T}_3}$ begins



Some magic

Example

Passing from $\underline{\mathbf{w}}$ to $\mathbf{w}$ by dropping the symbols "$\cdot$" and "$|$", we obtain the tree $\mathcal{T}_m$



Dendrodistinctivity

The existence of the inverse map from $\mathcal{W}_m^k$ to $\mathcal{X}_m^k$ is equivalent to the following theorem, which shows one can recover the partitioning of a word.

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Theorem (Dendrodistinctivity, H. Thomas)

Let $\mathcal{T}_m$ be the infinite complete m-ary tree obtained from $\underline{\mathcal{T}_m}$ by forgetting the partitionings. Then each word of length k on $\{0, 1, \ldots, m-1\}$ appears exactly once.

Thank You!