Financial Mathematics 5001 : Homework 11 (0034)

Due on 8 December 2010

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Solutions

0034 – 1. Let $Q(x, y) = 84x^2 - 24xy + 91y^2$.

a) Find a rotation matrix $R \in \mathbb{R}^{2\times 2}$ such that $Q \circ L_R$ is diagonal, i.e., such that $Q(L_R(x, y))$ has no mixed terms.

b) Graph $Q(x, y) = 100$.

a) Let $M = \begin{pmatrix} 84 & -12 \\ -12 & 91 \end{pmatrix}$ be the symmetric matrix corresponding to Q.

We first find the eigenvalues of M :

$$\det\begin{pmatrix} 84-\lambda & -12 \\ -12 & 91-\lambda \end{pmatrix} = 7500 - 175\lambda + \lambda^2 = (-100+\lambda)(-75+\lambda) = 0.$$

Therefore, $\lambda = 100$ or $\lambda = 75$.

Next, we find the eigenvectors corresponding to the two eigenvalues.

$\lambda = 100 : \begin{pmatrix} 84-100 & -12 \\ -12 & 91-100 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, so that this eigenspace is spanned by $v1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$\lambda = 75 : \begin{pmatrix} 84-75 & -12 \\ -12 & 91-75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, so that this eigenspace is spanned by $v2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Since we want a rotation matrix, need that v1 and v2 are orthonormal. It is clear that they are orthogonal, so we just have to normalize so that v1.v1 = v2.v2 = 1. Multiplying each by 1/5 does the trick.

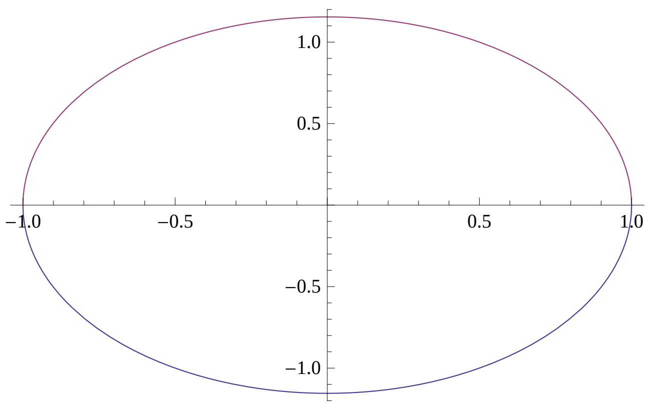
Then $R = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$. Observe that since $R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$, and $\det(R) = 1$,

R is a clockwise rotation of about $\arccos(3/5) = 0.927295$ radians, or about 53.13 degrees.

b) Applying the diagonalizing rotation, we have

$R^{-1} M R = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 84 & -12 \\ -12 & 91 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 100 & 0 \\ 0 & 75 \end{pmatrix}$. Thus,

we can graph Q by first graphing the diagonal form of Q,
and then rotating back. The diagonal form is $100x^2 + 75y^2 = 100$, which is an elipse :



Rotating back, we obtain the original graph of Q(x, y) = 100: