Financial Mathematics 5001 : Homework 12 (0035, 0036)

Due on 15 December 2010

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Solutions

0035 – 1. Let e1, e2, and e3 be the standard basis of $\mathbb{R}^{1\times 3}$.

a) Let v1 = (5, 3, -1) and v2 = (1, -2, -1). Find a rotation matrix R and scalars c1, c2, such that c1 e1 R = v1 and c2 e2 R = v2.

b) Find a rotation matrix L such that v1 L $\in \mathbb{R}$e1 and v2 L $\in \mathbb{R}$e2.

a) It is clear that since R is a rotation matrix (and therefore length – preserving),

we must take the first row of R to be $\left(\frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}, -\frac{1}{\sqrt{35}}\right)$ and the second row to be

$\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$. To find the third row, we use the fact that rotation matrices are orthonormal. Thus,

if the third row is (x1, x2, x3), then we have the system of three equations :

$5\,x1 + 3\,x2 - x3 = 0,$

$x1 - 2\,x2 - x3 = 0,$ and

$x1^2 + x2^2 + x3^2 = 1.$

The solution that gives R a determinant of one is $x1 = \sqrt{\frac{5}{42}}$, $x2 = -2\sqrt{\frac{2}{105}}$, and $x3 = \frac{13}{\sqrt{210}}$. Thus,

$$R = \begin{pmatrix} \sqrt{\frac{5}{7}} & \frac{3}{\sqrt{35}} & -\frac{1}{\sqrt{35}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{5}{42}} & -2\sqrt{\frac{2}{105}} & \frac{13}{\sqrt{210}} \end{pmatrix} \left(\text{and } c1 = \sqrt{35},\ c2 = \sqrt{6}\right).$$

b) We take $L = R^{-1}$, since c1 e1 R = v1 is equivalent to c1 e1 = v1 R^{-1},
so that v1 $R^{-1} \in \mathbb{R}$e1 (and likewise for v2, e2).

$$\text{Thus, } L = \begin{pmatrix} \sqrt{\frac{5}{7}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{5}{42}} \\ \frac{3}{\sqrt{35}} & -\sqrt{\frac{2}{3}} & -2\sqrt{\frac{2}{105}} \\ -\frac{1}{\sqrt{35}} & -\frac{1}{\sqrt{6}} & \frac{13}{\sqrt{210}} \end{pmatrix}.$$

0035 – 2. Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix}$. Find rotation matrices $K \in \mathbb{R}^{2\times2}$ and $L \in \mathbb{R}^{3\times3}$ such that KML is "diagonal".

$MM^t = \begin{pmatrix} 5 & 6 \\ 6 & 10 \end{pmatrix}$, which has eigenvalues 14 and 1 ,

with associated eigenvectors (2, 3) and (3, −2). Normalizing, let $K = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$.

On the other hand, $M^t M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 13 & 3 \\ 0 & 3 & 1 \end{pmatrix}$, which has eigenvalues 14,
1, and 0 with associated eigenvectors (2, 13, 3), (3, 0, −2),
and (2, −1, 3). Normalizing, let $L = \begin{pmatrix} \sqrt{\frac{2}{91}} & \frac{-3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \sqrt{\frac{13}{14}} & 0 & -\frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{14}} \end{pmatrix}$.

Then the singular value decompostiion theorem implies that KML = D, where $D = \begin{pmatrix} \sqrt{14} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

0036 – 1. Let $A = \begin{pmatrix} 1 & -2 & -5 \\ 2 & 2 & -4 \\ 0 & -2 & 1 \end{pmatrix}$. Let d = det A. Let B be the cofactor matrix of A. Let I be the 3×3 identity matrix.

a) Find B.

b) Find B^t.

c) Find AB^t.

d) Find $B^t A$.

e) Find dI.

a) $B = \begin{pmatrix} -6 & -2 & -4 \\ 12 & 1 & 2 \\ 18 & -6 & 6 \end{pmatrix}$.

b) $B^t = \begin{pmatrix} -6 & 12 & 18 \\ -2 & 1 & -6 \\ -4 & 2 & 6 \end{pmatrix}$.

c), d), e) $AB^t = B^t A = dI = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$.

0036 - 2. Suppose x, y, and z solve the system
$2x - 7y - 3z = 4$,
$7x + 3y + 6z = 3$, and
$9x + 5y - 2z = 1$.
Using Cramer' s rule, express z as a quotient of determinants. (No need to compute z.)

$$z = \frac{\det\begin{pmatrix} 2 & -7 & 4 \\ 7 & 3 & 3 \\ 9 & 5 & 1 \end{pmatrix}}{\det\begin{pmatrix} 2 & -7 & -3 \\ 7 & 3 & 6 \\ 9 & 5 & -2 \end{pmatrix}}.$$