Financial Mathematics 5001 : Homework 2 (0022 – 0023)

Due on 5 October 2011

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Solutions

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23 – 1. Compute

$$\begin{pmatrix} -1 & 2 & 0 & -9 \\ 5 & 2 & 4 & -1 \\ -3 & 6 & -4 & 9 \end{pmatrix} . \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 4 \\ -2 & 0 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} 15 & 3 & -18 \\ 13 & 9 & 14 \\ -23 & 9 & 2 \end{pmatrix}$$

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23 – 2. Compute

$$\begin{pmatrix} 0 & 3 & 6 & 9 \\ 3 & -2 & 9 & 12 \\ -1 & 8 & -3 & -4 \end{pmatrix} + \begin{pmatrix} -3 & 1 & -5 & 8 \\ 2 & 1 & -3 & 0 \\ 7 & 0 & 1 & 6 \end{pmatrix}.$$

$$\begin{pmatrix} -3 & 4 & 1 & 17 \\ 5 & -1 & 6 & 12 \\ 6 & 8 & -2 & 2 \end{pmatrix}$$

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23 – 3. Find the transpose of

$$\begin{pmatrix} -2 & -3 & -4 & -5 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 4 \\ -3 & 3 \\ -4 & 2 \\ -5 & 1 \end{pmatrix}$$

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23 – 4. Compute

$$\begin{pmatrix} -1 & 3 & 0 & -7 \\ 2 & -2 & 7 & 3 \\ 4 & 3 & -4 & 6 \end{pmatrix} \oplus \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 0 \\ 1 & 4 & 7 \end{pmatrix}.$$

$$\begin{pmatrix} -1 & 3 & 0 & -7 & 0 & 0 & 0 \\ 2 & -2 & 7 & 3 & 0 & 0 & 0 \\ 4 & 3 & -4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 7 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 7 \end{pmatrix}$$

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23 – 5. $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$.

Compute A⊗B and B⊗A.

$$A\otimes B = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 2 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 & -3 \\ 0 & -2 & 0 & 0 & 0 & -6 \end{pmatrix},\ B\otimes A = \begin{pmatrix} 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 4 & 2 & 2 \\ 0 & 0 & 0 & -2 & 0 & -6 \end{pmatrix}$$

23-6. Find the left conjugate of $\begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 6 & 8 \end{pmatrix}$ by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} . \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 6 & 8 \end{pmatrix} . \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} & -\frac{5}{3} \\ -14 & -9 & \frac{4}{3} \\ -12 & -9 & 8 \end{pmatrix}$$

23-7. Let $A = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix}$, and $C = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix}$.

a) Compute AB and BA.
b) Compute ACB.
c) Compute e^C.
d) Compute e^{ACB}.
e) Compute [2]⊕[3].
f) Compute A⊕C.
g) Compute A⊗C.

a) $AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} -41 & 18 \\ -120 & 52 \end{pmatrix}$

c) $\begin{pmatrix} e^4 & 0 \\ 0 & e^7 \end{pmatrix}$

d) $e^{ACB} = Ae^C B = \begin{pmatrix} 16 e^4 - 15 e^7 & -6 e^4 + 6 e^7 \\ 40 e^4 - 40 e^7 & -15 e^4 + 16 e^7 \end{pmatrix}$

e) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

f) $\begin{pmatrix} 2 & 3 & 0 & 0 \\ 5 & 8 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$

g) $\begin{pmatrix} 8 & 0 & 12 & 0 \\ 0 & 14 & 0 & 21 \\ 20 & 0 & 32 & 0 \\ 0 & 35 & 0 & 56 \end{pmatrix}$

23-8. Let $A = \begin{pmatrix} 4 & 6 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 1 \\ 0 & -7 \end{pmatrix}$.

a) Compute A+B.
b) Compute B+A.
c) Compute AB.
d) Compute BA.
e) Compute $A \oplus B$.
f) Compute $B \oplus A$.
g) Compute $A \otimes B$.
h) Compute $B \otimes A$.

a) $\begin{pmatrix} -1 & 7 \\ 2 & -6 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 7 \\ 2 & -6 \end{pmatrix}$

c) $\begin{pmatrix} -20 & -38 \\ -10 & -5 \end{pmatrix}$

d) $\begin{pmatrix} -18 & -29 \\ -14 & -7 \end{pmatrix}$

e) $\begin{pmatrix} 4 & 6 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & -7 \end{pmatrix}$

f) $\begin{pmatrix} -5 & 1 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 2 & 1 \end{pmatrix}$

g) $\begin{pmatrix} -20 & 4 & -30 & 6 \\ 0 & -28 & 0 & -42 \\ -10 & 2 & -5 & 1 \\ 0 & -14 & 0 & -7 \end{pmatrix}$

h) $\begin{pmatrix} -20 & -30 & 4 & 6 \\ -10 & -5 & 2 & 1 \\ 0 & 0 & -28 & -42 \\ 0 & 0 & -14 & -7 \end{pmatrix}$

23-9. Let $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_{11} = \begin{pmatrix} 6 & 7 \\ 8 & -7 \end{pmatrix}$, $A_{12} = \begin{pmatrix} -7 & 3 \\ 2 & 6 \end{pmatrix}$, $A_{21} = \begin{pmatrix} 6 & 5 \\ 3 & 0 \end{pmatrix}$, $A_{22} = \begin{pmatrix} 1 & 0 \\ -5 & 8 \end{pmatrix}$. Compute $(E_{11} \otimes A_{11}) + (E_{12} \otimes A_{12}) + (E_{21} \otimes A_{21}) + (E_{22} \otimes A_{22})$.

$\begin{pmatrix} 6 & 7 & -7 & 3 \\ 8 & -7 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 3 & 0 & -5 & 8 \end{pmatrix}$.

23-10. Let $B_{11} = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix}$, $B_{12} = \begin{pmatrix} 9 & 7 \\ 8 & 2 \end{pmatrix}$, $B_{21} = \begin{pmatrix} 6 & -2 \\ 5 & 9 \end{pmatrix}$,

$B_{22} = \begin{pmatrix} 2 & 7 \\ 4 & 8 \end{pmatrix}$. Find C_{11}, C_{12}, C_{21}, and C_{22} such that $(E_{11} \otimes C_{11}) + (E_{12} \otimes C_{12}) + (E_{21} \otimes C_{21}) + (E_{22} \otimes C_{22}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$.

Take $C_{ij} = B_{ij}$.

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23 – 11. Define $M : (\mathbb{R}^{2\times2})^{2\times2} \to \mathbb{R}^{2\times2}$ by the rule
$M((E_{11} \otimes X_{11}) + (E_{12} \otimes X_{12}) + (E_{21} \otimes X_{21}) + (E_{22} \otimes X_{22})) = (E_{11} X_{11}) + (E_{12} X_{12}) + (E_{21} X_{21}) + (E_{22} X_{22})$.
Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -5 & -6 \\ -7 & -8 \end{pmatrix}$. Compute $M(A \otimes B)$ and AB.

$M(A \otimes B) = AB = \begin{pmatrix} -19 & -22 \\ -43 & -50 \end{pmatrix}$.

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23 – 12. Let $M = \begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$ and define $t - \mathrm{cof}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

a) Compute $(t - \mathrm{cof}\ M)\ M$, $M\ (t - \mathrm{cof}\ M)$, $(\det M)\ I$.

b) Show that for all $K \in \mathbb{R}^{2\times2}$, $(t - \mathrm{cof}\ K)\ K = K (t - \mathrm{cof}\ K) = (\det K)\ I$.

a) $t\text{-}\mathrm{cof}\ M = \begin{pmatrix} 4 & -5 \\ -6 & 7 \end{pmatrix}$. $(t - \mathrm{cof}\ M)\ M = M\ (t - \mathrm{cof}\ M) = (\det M)\ I = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

b) Let $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $t\text{-}\mathrm{cof}\ K = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $(t - \mathrm{cof}\ K)\ K = K\ (t - \mathrm{cof}\ K) = (\det K)\ I = \begin{pmatrix} -b c + a d & 0 \\ 0 & -b c + a d \end{pmatrix}$.

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23 – 13. Let $M = \begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$, $X = \begin{pmatrix} 2 & 7 \\ -3 & -8 \end{pmatrix}$, and define $\mathrm{DET}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC$,
$T - \mathrm{COF}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$. Let $\mathcal{M} = M \otimes X$.

a) Compute $(T - \mathrm{COF}\ \mathcal{M})\ \mathcal{M}$, $\mathcal{M}\ (T - \mathrm{COF}\ \mathcal{M})$, $I \otimes [\mathrm{DET}\ \mathcal{M}]$.

b) Show that for all $K \in (\mathbb{R}X)^{2\times2}$, $(T - \mathrm{COF}\ K)\ K = K\ (T - \mathrm{COF}\ K) = I \otimes [\mathrm{DET} K]$.

a) $T - \mathrm{COF}\ \mathcal{M} = \begin{pmatrix} 4X & -5X \\ -6X & 7X \end{pmatrix}$. $(T - \mathrm{COF}\ \mathcal{M})\ \mathcal{M} = \mathcal{M}\ (T - \mathrm{COF}\ \mathcal{M}) = I \otimes [\mathrm{DET}\ \mathcal{M}] = \begin{pmatrix} -2X^2 & 0 \\ 0 & -2X^2 \end{pmatrix}$.

b) Let $K = \begin{pmatrix} aX & bX \\ cX & dX \end{pmatrix}$, so that $T - \mathrm{COF}\ K =$
$\begin{pmatrix} dX & -bX \\ -cX & aX \end{pmatrix}$. Then $(T - \mathrm{COF}\ K)\ K = K\ (T - \mathrm{COF}\ K) = I \otimes [\mathrm{DET} K] = \begin{pmatrix} (-b c + a d)\ X^2 & 0 \\ 0 & (-b c + a d)\ X^2 \end{pmatrix}$.
