Financial Mathematics 5001: Homework 2 (0004-0010)

Due on 22 September 2010

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Solutions

0004-1

Let $\alpha : \{1,2,3\} \to \{1,2,3\}$ be defined by $\alpha(1) = 2, \alpha(2) = 1, \alpha(3) = 3$ and let $\beta : \{1,2,3\} \to \{1,2,3\}$ be defined by $\beta(1) = 1, \beta(2) = 3, \beta(3) = 2$. Check that $\alpha \circ \beta \neq \beta \circ \alpha$.

$(\alpha \circ \beta)(1) = \alpha(\beta(1)) = \alpha(1) = 2$, while $(\beta \circ \alpha)(1) = \beta(\alpha(1)) = \beta(2) = 3$. Thus, the two functions cannot be equal since they disagree on the same input.

0004-2

Let $A := \{2,4,6,8,9\}$ and let $B := \{1,3,5\}$. The set of functions from A to B is denoted B^A. Compute $\#(B^A)$.

Each element of A can be mapped to any element of B. Since $\#(B) = 3$ and $\#(A) = 5$, there are 3^5 such functions.

0005-1

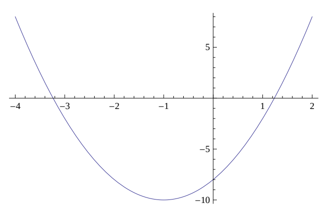
Eliminate the linear term in $y = -(x^2/2) - 7x + 2$.

$-(x^2/2) - 7x + 2 = -1/2(x^2 + 14x + 49 - 49 - 4) = -1/2((x+7)^2 - 53) = -1/2(x+7)^2 + 53/2.$

0005-2

Let $f(x) = 2x^2 + 4x - 8$.

1. Eliminate the linear term in $y = f(x)$.
2. Graph $y = f(x)$. Indicate the coordinates of the lowest point on the graph.

1. $2x^2 + 4x - 8 = 2(x^2 + 2x - 4) = 2(x^2 + 2x + 1 - 5) = 2((x+1)^2 - 5) = 2(x+1)^2 - 10.$
2. 

   The lowest point on the graph is $(-1, -10)$.

0006-1

Compute $\binom{9}{6}$.

$\binom{9}{6} = \frac{9!}{6!3!} = \frac{9 \cdot 8 \cdot 7}{6} = 3 \cdot 4 \cdot 7 = 84.$

0006-2

Write $\binom{k}{4}$ as a polynomial in k. Compute each coefficient of that polynomial.

$\binom{k}{4} = \frac{k!}{4!(k-4)!} = \frac{k(k-1)(k-2)(k-3)}{24} = \frac{1}{24}k^4 + \frac{-1}{4}k^3 + \frac{11}{24}k^2 + \frac{-1}{4}k.$

0006-3

What is the degree of $\binom{n}{125}$ as a polynomial in n?

$\binom{n}{125} = \frac{n(n-1)\cdots(n-125+1)}{125!}$. Since the maximum degree that results when expanding occurs when n is selected from each of the terms $(n-i)$, $0 \le i \le 124$, and since there are 125 such terms, the degree of $\binom{n}{125}$ is 125.

0007-1

Write the first four terms in the binomial expansion of $(p+q)^9$.

$(p+q)^9 = p^9 + \binom{9}{1}p^8q + \binom{9}{2}p^7q^2 + \binom{9}{3}p^6q^3 + \dots.$

0008-1

How many monomials are there of degree ≤ 8 in three variables?

$\binom{8+3}{3} = \binom{11}{3} = 165.$

0008-2

How many monomials are there of degree $= 8$ in four variables?

$\binom{8+4-1}{4-1} = \binom{11}{3} = 165.$ The reason this is the same as above is that for each monomial of degree ≤ 8 in three variables, there is a unique monomial of degree $= 8$ in four variables obtained by multipling by the fourth variable until the degree is 8.

0008-3

How many monomials are there of degree ≤ 4 in seven variables?

$\binom{7+4}{4} = \binom{11}{4} = 330.$

0008-4

How many monomials are there of degree ≤ 3 in seven variables?

$\binom{7+3}{3} = \binom{10}{3} = 120.$

0008-5

How many monomials are there of degree = 4 in seven variables?

$\binom{4+7-1}{7-1} = \binom{10}{6} = 210.$

0009-1

$$\frac{d}{dx}\left[\frac{x^5 e^{-x^3}}{4(\cos(x))(\tan(5))}\right].$$

$$\frac{1}{4(\tan(5))}\left(5e^{-x^3}x^4\sec(x) - 3e^{-x^3}x^7\sec(x) + e^{-x^3}x^5\sec(x)\tan(x)\right).$$

0009-2

$$\frac{d}{dq}\left[(\ln(q))^{6+\sin(q)}\right].$$

$$(\ln(q))^{6+\sin(q)}\left(\cos(q)\ln(\ln(q)) + \frac{6+\sin(q)}{q\ln(q)}\right).$$

0009-3

$$\frac{d}{dt}\left[(t^5)(\sin^2(8^6))\right].$$

$$5(\sin^2(8^6))t^4.$$

0009-4

$$\frac{d}{dx}\left[(\cos(x))^{\csc(x)}\right].$$

$$\cos(x)^{\csc(x)}\left(-\cot(x)\csc(x)\ln(\cos(x)) - \sec(x)\right).$$

00010-1

Let $f(x) = |x|$.

1. Explain why the lack of a solution to

$$f'(c) = \frac{f(2) - f(-2)}{2-(-2)}, c \in (-2, 2)$$

does not contradict the Mean Value Theorem.

00010-1 continued on next page...

2. How many solutions are there to

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}, c \in (1, 2)?$$

1. We seek solutions c in $(-2, 2)$ to $f'(c) = 0$. But since $f(x) = |x|$, $f'(x) = 1$ for $x > 0$, $f'(x) = -1$ for $x < 0$, and $f'(0)$ isn't defined—therefore, there are no solutions. This does not contradict the Mean Value Theorem because f' is not defined on the interval $(-2, 2)$.

2. We seek solutions c in $(1, 2)$ to $f'(c) = 1$. But, as remarked above, for all $c \in (1, 2)$, $f'(c) = 1$. Therefore, there are an infinite number (with cardinality c) of solutions.

00010-2

Let $f(x) = x^3 + 3x^2 - 24x - 0$. Find the (maximal) intervals of increase and decrease for f.

$f'(x) = 3x^2 + 6x - 24 = 3(x - 2)(x + 4)$. Therefore, the sign of the derivative (which determines increase or decrease) can only possibly change at $x = 2$ and $x = -4$, and so the intervals of increase and decrease are $(-\infty, -4)$, $(-4, 2)$, $(2, \infty)$. A simple check yields that f increases on $(-\infty, -4)$ and $(2, \infty)$, and decreases on $(-4, 2)$.