Financial Mathematics 5001 : Homework 4 (0026 –0027)

Due on 2 November 2011

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Solutions

0026 –1. Determine if (–1, 1, –1, 2, 8) is in the span of
(0, 1, –1, 5, –3),
(1, 8, –7, 38, –9),
(5, –2, 2, 1, 5),
(–3, 6, –5, 19, 16).

Performing elementary row operations on the matrix (which does not change the span)

$$\begin{pmatrix} 0 & 1 & -1 & 5 & -3 \\ 1 & 8 & -7 & 38 & -9 \\ 5 & -2 & 2 & 1 & 5 \\ -3 & 6 & -5 & 19 & 16 \end{pmatrix}$$

we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 85 \\ 0 & 0 & 1 & 0 & -367 \\ 0 & 0 & 0 & 1 & -91 \end{pmatrix}.$$

But
$(-1, 1, -1, 2, -5) \neq -(1, 0, 0, 0, 200) + (0, 1, 0, 0, 85) - (0, 0, 1, 0, -367) + 2(0, 0, 0, 1, -91)$,
and so $(-1, 1, -1, 2, -5)$ is not in the span of the four original vectors.

0026 –2. Let $S \subseteq \mathbb{R}^5$ be the span of
(8,2,9,2,1),
(5,–7,–6,8,–2),
(6,–4,–1,6,–1),
(7,–1,4,4,0),
(0,8,1,–1,2),
(6,4,0,5,1).
Extract a basis of *S* from these six vectors. Your answer should be a subset of the set of these six vectors.

$(6, -4, -1, 6, -1) = 1/3\,(8, 2, 9, 2, 1) + 2/3\,(5, -7, -6, 8, -2)$
$(7, -1, 4, 4, 0) = 2/3\,(8, 2, 9, 2, 1) + 1/3\,(5, -7, -6, 8, -2)$
$(6, 4, 0, 5, 1) = 1/3\,(8, 2, 9, 2, 1) + 2/3\,(5, -7, -6, 8, -2) + (0, 8, 1, -1, 2)$
It remains to check that $(8, 2, 9, 2, 1)$, $(5, -7, -6, 8, -2)$, and $(0, 8, 1, -1, 2)$ are linearly independent.

Using row operations on

$$\begin{pmatrix} 8 & 2 & 9 & 2 & 1 \\ 5 & -7 & -6 & 8 & -2 \\ 0 & 8 & 1 & -1 & 2 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} 1 & 0 & 0 & \frac{197}{226} & \frac{1}{226} \\ 0 & 1 & 0 & -\frac{13}{226} & \frac{55}{226} \\ 0 & 0 & 1 & -\frac{61}{113} & \frac{6}{113} \end{pmatrix}.$$

Therefore, the required basis is (8, 2, 9, 2, 1), (5, −7, −6, 8, −2), and (0, 8, 1, −1, 2) .

0026−3. Are the vectors
(2, 4, 6, 8),
(1, 2, 5, −1),
(3, 6, 4, −6),
(3, 6, −7, −9)
linearly independent? If not, express one as a linear combination of the others.

They aren't :
−3 (1, 2, 5, −1) +2(3, 6, 4, −6) = (3, 6, −7, −9).

0026−4. Find the image of $M = \begin{pmatrix} 0 & 1 & -1 & 5 & -3 \\ 1 & 8 & -7 & 38 & -9 \\ 5 & -2 & 2 & 1 & 5 \\ -3 & 6 & -5 & 19 & 16 \end{pmatrix}$.

We use column operations to reduce M to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the image is all of $\mathbb{R}^4$.

0026−5. Find the dimension of the kernel of

$$M = \begin{pmatrix} 0 & 1 & -1 & 5 & -3 \\ 1 & 8 & -7 & 38 & -9 \\ 5 & -2 & 2 & 1 & 5 \\ -3 & 6 & -5 & 19 & 16 \end{pmatrix}.$$

From 0026−4, we have that dim (im (M)) = 4. Since dim (ker (M)) + dim (im (M)) = 5, dim (ker (M)) = 1.

0026−6.

a) For each of the following two matrices, compute the dimension of its kernel and the dimension of its image.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -3 \\ 3 & 4 & -7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 4 & -3 \end{pmatrix}$$

b) For each of the following two matrices, compute the dimension of its kernel and the dimension of its image.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -3 \\ 3 & 4 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -3 \\ 1 & 1 & 10 \end{pmatrix}$$

a) The hint shows us that the two matrices are related by an elementary column operation
so that the dimensions of their kernels are equal and the dimensions of their images are equal.

We row reduce to obtain $\begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$, so that the dimension of each kernel is 1 and the dimension of each image is 2.

b) The two matrices are now related by an elementary row operation,
but we again conclude that the dimensions of their kernels are equal and the dimensions of their images are equal.

We row reduce to obtain $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so that the dimension of each kernel is 0 and the dimension of each image is 3.

0026 –7.

a) Determine which of these two matrices is invertible :

$\begin{pmatrix} 6 & 3 & 4 \\ 2 & 2 & 3 \\ 8 & -1 & -3 \end{pmatrix}, \begin{pmatrix} 6 & 3 & 4 \\ 2 & 2 & 3 \\ 7 & -1 & -3 \end{pmatrix}$.

b) Invert it.

a) We row reduce $\begin{pmatrix} 6 & 3 & 4 \\ 2 & 2 & 3 \\ 8 & -1 & -3 \end{pmatrix}$ to obtain $\begin{pmatrix} 1 & 0 & -\frac{1}{6} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{pmatrix}$. Since the dimension of the kernel is nonzero,

this matrix is not invertible.

On the other hand, row reducing $\begin{pmatrix} 6 & 3 & 4 \\ 2 & 2 & 3 \\ 7 & -1 & -3 \end{pmatrix}$ gives the identity so that it is invertible.

b) $\begin{pmatrix} 3 & -5 & -1 \\ -27 & 46 & 10 \\ 16 & -27 & -6 \end{pmatrix}$

0026 –8. Solve
$2x + 2y + 5z = p$
$5x + 3y + 7z = q$
$8x + 3y + 6z = r$,
where p, q, and r are arbitrary.

We encode the system of equation in the matrix

$$\begin{pmatrix} 2 & 2 & 5 & p \\ 5 & 3 & 7 & q \\ 8 & 3 & 6 & r \end{pmatrix}$$

We then row reduce to obtain the identity in the leftmost three columns

$$\begin{pmatrix} 1 & 0 & 0 & -3p + 3q - r \\ 0 & 1 & 0 & 26p - 28q + 11r \\ 0 & 0 & 1 & -9p + 10q - 4r \end{pmatrix}$$

Therefore, $x = -3p + 3q - r$, $y = 26p - 28q + 11r$, $z = -9p + 10q - 4r$.

0027 – 1. Write $\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$ as a product of elementary matrices.

We perform the following operations to write $\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$ in fully canonical form : R1 = R1 * 1/2, R2 = R2 − 3 R1, R2 = R2 * 1/3, R1 = R1 − 2 R2. In matrices, we have that

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Multiplying by inverses, we obtain

$$\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

0027 − 2. Find det $\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$.

$$\det\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 2 * 1 * 3 * 1 = 6.$$

0027 − 3. Find the signed area of the oriented parallelogram ((2, 3), (4, 9)).

From class, we know that the determinant gives us this signed area. Therefore, the answer is 6.

0027 − 4. Write $\begin{pmatrix} 2 & 4 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ as a product of elementary matrices.

We can perform the same operations as in 0027 − 1 to get to the matrix

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Performing R3 = R3 * −1/2 gets us to fully canonical form. In matrices, we have that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplying by inverses, we obtain

$$\begin{pmatrix} 2 & 4 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

0027 − 5. Find det $\begin{pmatrix} 2 & 4 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

$$\det\begin{pmatrix} 2 & 4 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 2 * 1 * 3 * 1 * -2 = -12.$$

0027 − 6. Find the signed area of the oriented parallelogram ((2, 3, 0), (4, 7, 0), (0, 0, −2)).

From class, we know that the determinant gives us this signed area. Therefore, the answer is −12.

0027 − 7. Write $\begin{pmatrix} 3 & 1 & 2 \\ 9 & 7 & 2 \\ 4 & 2 & 2 \end{pmatrix}$ as a product of elementary matrices, then a fully canonical matrix, then more elementary matrices.

We perform the following operations to write $\begin{pmatrix} 3 & 1 & 2 \\ 9 & 7 & 2 \\ 4 & 2 & 2 \end{pmatrix}$ in fully canonical form : R1 = R1/3, R2 = R2 − 9 R1,

R3 = R3 − 4 R1, R2 = R2/4, R1 = R1 − 1/3 R2, R3 = R3 − 2/3 R2, C3 = C3 − C1, C3 = C3 + C2. In matrices, we have that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix}.\begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -9 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 3 & 1 & 2 \\ 9 & 7 & 2 \\ 4 & 2 & 2 \end{pmatrix}.\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Multiplying by inverses, we obtain

$$\begin{pmatrix} 3 & 1 & 2 \\ 9 & 7 & 2 \\ 4 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

0027−8. Find $\det \begin{pmatrix} 3 & 1 & 2 \\ 9 & 7 & 2 \\ 4 & 2 & 2 \end{pmatrix}$.

$$\det \begin{pmatrix} 3 & 1 & 2 \\ 9 & 7 & 2 \\ 4 & 2 & 2 \end{pmatrix} = \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0, \text{ since } \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

0027−9. Find the signed area of the oriented parallelogram ((3, 9, 4), (1, 7, 2), (2, 2, 2)).

$\det \begin{pmatrix} 3 & 9 & 4 \\ 1 & 7 & 2 \\ 2 & 2 & 2 \end{pmatrix}$=0, since (3, 9, 4)= (1, 7, 2)+ (2, 2, 2). From class, we know that the determinant gives us this signed area. Therefore, the answer is 0.