Financial Mathematics 5001 : Homework 5 (0028 –0030)

Due on 9 November 2011

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Solutions

0028–1. Compute $\det\begin{pmatrix} 6 & 2 & -3 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix}$.

$\det\begin{pmatrix} 6 & 2 & -3 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix} =$

-135. (In anticipation of the next problem, do this by Laplacian determinant expansion by minors with the first row.)

0028–2. Compute $\det\begin{pmatrix} 2 & 4 & -8 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix}$.

$\det\begin{pmatrix} 2 & 4 & -8 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix} = 12$. (Since only the first row has changed, we can reuse the minors from 0028 – 1.)

0028–3. Compute $\det\begin{pmatrix} 6+2 & 2+4 & -3-8 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix}$.

$$\det\begin{pmatrix} 6+2 & 2+4 & -3-8 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix} = \det\begin{pmatrix} 6 & 2 & -3 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix} + \det\begin{pmatrix} 2 & 4 & -8 \\ 4 & 5 & -9 \\ -7 & 1 & -9 \end{pmatrix} = -135 + 12 = -123.$$

0028–4. Compute $\det\begin{pmatrix} 6 & 9 & 5 & 4 & 1 \\ 0 & -2 & 5 & 9 & -1 \\ 0 & 0 & 1 & -2 & -7 \\ 0 & 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

The determinant of an uppertraianglular matrix is just the product of its diagonal entries. Therefore, the answer is (6)(–2)(–2)=24.

0028–5. Suppose x, y, and z solve the system

$$\begin{aligned} 2x - 7y - 3z &= 7 \\ 7x + 3y + 6z &= 3 \\ 9x + 5y - 2z &= 2. \end{aligned}$$

Using Cramer's rule, express y as a quotient of determinants. (No need to compute y.)

$$y = \begin{pmatrix} 2 & 7 & -3 \\ 7 & 3 & 6 \\ 9 & 2 & -2 \end{pmatrix}.$$

0029 -1. Let $B : \mathbb{R}^5 \times \mathbb{R}^5 \longrightarrow \mathbb{R}$ be the bilinear form defined by

$$[B] = \begin{pmatrix} 4 & 7 & 9 & 0 & 6 \\ 5 & 0 & 8 & 0 & 2 \\ 1 & -2 & -4 & 1 & -8 \\ 4 & -2 & 3 & 3 & 1 \\ 0 & 4 & 0 & 9 & 0 \end{pmatrix}.$$

a. Let $v = (-1, 3, 1, 0, 0)$, $w = (1, -1, 0, 1, 0)$. Compute $B(v, w)$.

b. Define $Q : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ by $Q(v) = B(v, v)$. Write out $Q(p, q, r, s, t)$.

c. Find a symmetric matrix $M \in \mathbb{R}^{5\times5}$ such that if S is the SBF defined by $[S] = M$, then $S(v, v) = B(v, v)$.

a. $B(v, w) = (-1\ 3\ 1\ 0\ 0) . \begin{pmatrix} 4 & 7 & 9 & 0 & 6 \\ 5 & 0 & 8 & 0 & 2 \\ 1 & -2 & -4 & 1 & -8 \\ 4 & -2 & 3 & 3 & 1 \\ 0 & 4 & 0 & 9 & 0 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 22$

b. $Q(p, q, r, s, t) = 4p^2 + 12pq + 10pr + 6qr - 4r^2 + 4ps - 2qs + 4rs + 3s^2 + 6pt + 6qt - 8rt + 10st$

c. We can read this matrix off from the coefficients of $Q(p, q, r, s, t)$:

$$M = \begin{pmatrix} 4 & 6 & 5 & 2 & 3 \\ 6 & 0 & 3 & -1 & 3 \\ 5 & 3 & -4 & 2 & -4 \\ 2 & -1 & 2 & 3 & 5 \\ 3 & 3 & -4 & 5 & 0 \end{pmatrix}.$$

0029-2. Let Q: $\mathbb{R}^5 \to \mathbb{R}$ be the quatric form defined by
$Q(p,q,r,s,t) = 2p^2 + 4q^2 - 7r^2 - 9s^2 + 4pq - 8pr - 6ps - 2pt - 6qr - 8qs + 4qt - 6rs + 2rt + 100st$.
Let B: $\mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ be the polarization of Q. Write out the matrix [B] of B.

$$\begin{pmatrix} 2 & 2 & -4 & -3 & -1 \\ 2 & 4 & -3 & -4 & 2 \\ -4 & -3 & -7 & -3 & 1 \\ -3 & -4 & -3 & -9 & 50 \\ -1 & 2 & 1 & 50 & 0 \end{pmatrix}.$$

0029-3. Let $Q : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form defined by $Q(x, y) = 2x^2 + 6xy + y^2$. Determine whether Q is positive semidefinite.

It isn't: $Q(-3/2, 1) = (3/2)^2 - 6(3/2) + 1 = -7/2 < 0$.

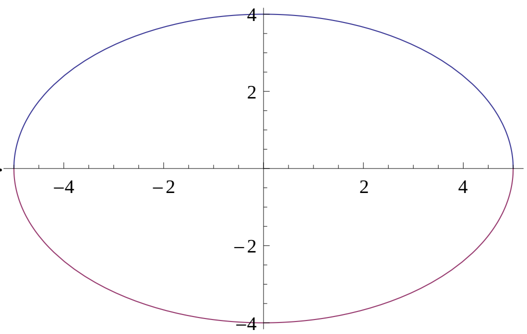
0029-4. Let $P : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form defined by $P(x, y) = \frac{x^2}{25} + \frac{y^2}{16}$.

a. Graph $\{(x, y) \mid P(x, y) = 1\}$.

b. Let $v = (2, 8)$. Let B be the polarization of P.

Find a vector $w \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $B(v, w) = 0$.

a.



b. $[B] = \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$. Then $B(v, w) = B((2, 8), (w1, w2)) = \frac{2\,w1}{25} + \frac{w2}{2}$. Thus, $B(v, w) = 0$ if $w = \left(a, -\frac{4\,a}{25}\right)$, with $a \neq 0$.

0030–1. Let $B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the polarization of the quadratic form $Q: \mathbb{R}^2 \to \mathbb{R}$ defined by $Q(x, y) = 2\,x^2 + 6\,y^2$.

a. Compute $B((1, 0), (1, 0))$.

b. Compute $B((1, 0), (0, 1))$.

c. Compute $B((0, 1), (1, 0))$.

d. Compute $B((0, 1), (0, 1))$.

e. Let S be the set of all pairs $(v, w) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $Q(v) \leq 2$ and $Q(w) \leq 8$. Find $\max\{B(v, w) \mid v, w \in S\}$.

a. $[B] = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$. Thus, $B((1, 0), (1, 0)) = 2$.

b. $B((1, 0), (0, 1)) = 0$.

c. $B((0, 1), (1, 0)) = 0$.

d. $B((0, 1), (0, 1)) = 6$.

e. By Cauchy – Schwarz, $\left|B(v, w)\right| \leq \sqrt{Q(v)}\,\sqrt{Q(w)} = 4$. This is realized when (for example) $v = (1, 0)$ and $w = (2, 0)$.

0030–2. Let $R: \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form defined by $R(x, y, z) = 3\,x^2 + 2\,y^2 + z^2$.

Let $B: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the polarization of R.

Let $D = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, R(x, y, z) \leq 20\right\}$.

Let $v0 = (1, 2, -3)$.

Find $w0 \in D$ such that $B(v0, w0) = \max\{B(v0, w) \mid w \in D\}$.

By Cauchy – Schwarz, $\left|B(v0, w)\right| \leq \sqrt{R(v0)}\,\sqrt{R(w)} \leq 20$. On the other hand,
$B(v0, w) = B((1, 2, -3), (w1, w2, w3)) = 3\,w1 + 4\,w2 - 3\,w3$. Thus, we could take w to be, for example, $(1, 2, -3)$.
