Financial Mathematics 5001 : Homework 6 (0031 – 0032)

Due on 23 November 2011

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Solutions

0031 – 1. Find a 4×4 rotation matrix whose first column has entries $-\frac{1}{\sqrt{7}}, \sqrt{\frac{2}{7}}, \frac{1}{\sqrt{7}}, \sqrt{\frac{3}{7}}$.

Using Gram – Schmidt, we obtain

$$\begin{pmatrix} -\frac{1}{\sqrt{7}} & \sqrt{\frac{2}{35}} & \frac{1}{2\sqrt{5}} & \frac{\sqrt{3}}{2} \\ \sqrt{\frac{2}{7}} & \sqrt{\frac{5}{7}} & 0 & 0 \\ \frac{1}{\sqrt{7}} & -\sqrt{\frac{2}{35}} & \frac{2}{\sqrt{5}} & 0 \\ \sqrt{\frac{3}{7}} & -\sqrt{\frac{6}{35}} & -\frac{\sqrt{\frac{3}{5}}}{2} & \frac{1}{2} \end{pmatrix}$$

0031 – 2. Find a 2×2 rotation matrix M such that $L_M\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

These vectors lie on the unit circle. The first corresponds to the angle $\frac{2\pi}{3}$, the second to $-\frac{\pi}{4}$. Thus,

the angle of rotation is $-\frac{11\pi}{12}$. This corresponds to the rotation matrix $\begin{pmatrix} \mathrm{Cos}[-\frac{11\pi}{12}] & -\mathrm{Sin}[-\frac{11\pi}{12}] \\ \mathrm{Sin}[-\frac{11\pi}{12}] & \mathrm{Cos}[-\frac{11\pi}{12}] \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} \\ -\frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$.

0031 – 3.

a. Compute $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b. Show that the rows of R are orthonormal.

c. Compute R^{-1}.

a. $R = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3\sqrt{3}}{10} & \frac{2\sqrt{3}}{5} & -\frac{1}{2} \\ \frac{3}{10} & \frac{2}{5} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

b. R is a rotation matrix because it is the product of two rotation matrices, and so has orthonomal rows.

c. $R^{-1} = R^T = \begin{pmatrix} \frac{4}{5} & \frac{3\sqrt{3}}{10} & \frac{3}{10} \\ -\frac{3}{5} & \frac{2\sqrt{3}}{5} & \frac{2}{5} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

0032-1. Let $M = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$.

a) Find the eigenvalues of M.

b) For each eigenvalue, find an eigenvector with that eigenvalue.

c) Find a 2×2 invertible matrix C such that $C^{-1} M C$ is diagonal.

d) Find e^M.

a) We compute $\det(M - \lambda I) = \det\begin{pmatrix} -1-\lambda & 2 \\ 3 & -2-\lambda \end{pmatrix} =$

$-4 + 3\lambda + \lambda^2 = (-1+\lambda)(4+\lambda)$. λ is an eigenvalue of M when $\det(M - \lambda I) = 0$, so that $\lambda = -4$ or $\lambda = 1$.

b) $\lambda = -4$: $(M - \lambda I)\, v = (M + 4I)\, v = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} v1 \\ v2 \end{pmatrix} = 0$ if $3\,v1 + 2\,v2 = 0$. Therefore,

$\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is an eigenvector (with eigenvalue -4).

$\lambda = 1$: $(M - \lambda I)\, v = (M - I)\, v = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v1 \\ v2 \end{pmatrix} = 0$ if $-2\,v1 + v2 = 0$. Therefore, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector (with eigenvalue 1).

c) We can take C to have the eigenvectors of M as columns :

$C = \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix}$. Then $C^{-1} M C = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$.

d) $C C^{-1} e^M C C^{-1} = C e^{C^{-1} M C} C^{-1} = C e^{\begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}} C^{-1} = C \begin{pmatrix} e^{-4} & 0 \\ 0 & e \end{pmatrix} C^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{-4} & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{2}{5e^4} + \frac{3e}{5} & -\frac{2}{5e^4} + \frac{2e}{5} \\ -\frac{3}{5e^4} + \frac{3e}{5} & \frac{3}{5e^4} + \frac{2e}{5} \end{pmatrix}$.

0032-2. Let $M = \begin{pmatrix} 45 & -25 & -24 \\ 0 & -4 & 0 \\ 80 & -41 & -43 \end{pmatrix}$.

a) Find the eigenvalues of M.

b) For each eigenvalue λ of M, find a basis for $\ker(M - \lambda I)$, the λ-eigenspace of M.

c) Find a matrix C such that $C M C^{-1}$ is diagonal.

d) Find e^{tM}.

a) We compute (expanding using the second row) $\det(M - \lambda I) = \det\begin{pmatrix} 45-\lambda & -25 & -24 \\ 0 & -4-\lambda & 0 \\ 80 & -41 & -43-\lambda \end{pmatrix} = 60 + 23\lambda - 2\lambda^2 - \lambda^3 =$

$-(-5+\lambda)(3+\lambda)(4+\lambda)$. λ is an eigenvalue of M when $\det(M - \lambda I) = 0$, so that $\lambda = 5$, $\lambda = -4$, or $\lambda = -3$.

b) $\lambda = 5 : (M - \lambda I)\, v = (M - 5 I)\, v = \begin{pmatrix} 40 & -25 & -24 \\ 0 & -9 & 0 \\ 80 & -41 & -48 \end{pmatrix}\begin{pmatrix} v1 \\ v2 \\ v3 \end{pmatrix} = 0$ if $v2 = 0$,

$40\, v1 + 24\, v3 = 0$. Therefore, $\begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ is an eigenvector (with eigenvalue 5).

$\lambda = -4 : (M - \lambda I)\, v = (M + 4 I)\, v = \begin{pmatrix} 49 & -25 & -24 \\ 0 & 0 & 0 \\ 80 & -41 & -39 \end{pmatrix}\begin{pmatrix} v1 \\ v2 \\ v3 \end{pmatrix} = 0$ if $49\, v1 - 25\, v2 - 24\, v3 = 0$ and $80\, v1 - 41\, v2 - 39\, v3$. Therefore,

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector (with eigenvalue -4).

$\lambda = -3 : (M - \lambda I)\, v = (M + 3 I)\, v = \begin{pmatrix} 48 & -25 & -24 \\ 0 & -1 & 0 \\ 80 & -41 & -40 \end{pmatrix}\begin{pmatrix} v1 \\ v2 \\ v3 \end{pmatrix} = 0$ if $c2 = 0$,

$48\, v1 - 24\, v2 = 0$. Therefore, $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ is an eigenvector (with eigenvalue -3).

c) We can take C^{-1} to have the eigenvectors of M as columns :

$C^{-1} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 5 & 1 & 2 \end{pmatrix}$. Then $C = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ -5 & 2 & 3 \end{pmatrix}$, and $C M C^{-1} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

d) $C^{-1} C e^{M} C^{-1} C = C^{-1} e^{C M C^{-1}} C = C^{-1} e^{\begin{pmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix}} C =$

$$C^{-1}\begin{pmatrix} e^5 & 0 & 0 \\ 0 & e^{-4} & 0 \\ 0 & 0 & e^{-3} \end{pmatrix} C = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 5 & 1 & 2 \end{pmatrix}\begin{pmatrix} e^5 & 0 & 0 \\ 0 & e^{-4} & 0 \\ 0 & 0 & e^{-3} \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ -5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{e^3} + 6 e^5 & \frac{1}{e^4} + \frac{2}{e^3} - 3 e^5 & \frac{3}{e^3} - 3 e^5 \\ 0 & \frac{1}{e^4} & 0 \\ -\frac{10}{e^3} + 10 e^5 & \frac{1}{e^4} + \frac{4}{e^3} - 5 e^5 & \frac{6}{e^3} - 5 e^5 \end{pmatrix}.$$