Financial Mathematics 5001 : Homework 7 (0025 - 0026)

Due on 3 November 2010

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Solutions

0025 – 1. Let $M = \begin{pmatrix} 1 & 3 & 4 & -6 & 7 \\ 2 & 1 & 3 & -2 & 4 \\ -2 & 0 & -2 & 0 & -2 \\ -6 & 2 & -4 & 6 & -2 \end{pmatrix}$.

a) Do "row magic" to M, indicating clearly all the row operations.

b) Put M in row canonical form, indicating clearly all the row operations.

c) Put M in fully canonical form, indicating clearly all the row and column operations.

d) Write $M = E_1 \ldots \mathrm{E}_k \, C \, E_1' \ldots E_l'$ where $E_1 \ldots \mathrm{E}_k, E_1' \ldots E_l'$ are elementary and C is fully canonical.

a) R2 = R2 – 2 R1 , R3 = R3 + 2 R1, R4 = R4 + 6 R1,

$$\begin{pmatrix} 1 & 3 & 4 & -6 & 7 \\ 0 & -5 & -5 & 10 & -10 \\ 0 & 6 & 6 & -12 & 12 \\ 0 & 20 & 20 & -30 & 40 \end{pmatrix}$$

R2 = R2 / – 5, R1 = R1 – 3 R2, R3 = R3 – 6 R2, R4 = R4 – 20 R2

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \end{pmatrix}$$

R4 = R4 / 10, R2 = R2 + 2 R4

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

b) Continuing from part a, R3 ↔ R4

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

c) Continuing from part b, C3 = C3 – C1, C5 = C5 – C1

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

C3 = C3 – C2, C5 = C5 – 2 C2

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

C3 ↔ C4

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

d) Ri = Ri + kRj is given by the identity matrix with a k in row i, column j, multiplied on the left. Ci = Ci + kCj is given by the identity matrix with a k in row i, column j, multiplied on the right. Swapping rows or columns is done with the identity matrix where the 1 in row i column i and the 1 in row j column j now appear in row i column j and row j column i.

0025 - 2. Solve
w + x + y + 5 z = 5
2 w + 8 x + 8 y - 3 z = -9
4 w + 2 x + 2 y + 12 z = 14
-3 w + 5 x + 4 y + 2 z = -25

We form the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 5 & 5 \\ 2 & 8 & 8 & -3 & -9 \\ 4 & 2 & 2 & 12 & 14 \\ -3 & 5 & 4 & 2 & -25 \end{pmatrix}$$

and use elementary operations to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -20 \\ 0 & 0 & 1 & 0 & 19 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Therefore, w = 1, x = –20, y = 19, and z = 1.

0026 - 1. Determine if (-1, 1, -1, 2, -5) is in the span of
(0, 1, -1, 5, -7),
(1, 8, -7, 38, -65),
(5, -2, 2, 5, 4),
(-3, 6, -5, 19, -34).

Performing elementary row operations on the matrix (which does not change the span)

$$\begin{pmatrix} 0 & 1 & -1 & 5 & -7 \\ 1 & 8 & -7 & 38 & -65 \\ 5 & -2 & 2 & 5 & 4 \\ -3 & 6 & -5 & 19 & -34 \end{pmatrix}$$

we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -11 \\ 0 & 1 & 0 & 0 & -14 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

But
$(-1, 1, -1, 2, -5) = -(1, 0, 0, 0, -11) + (0, 1, 0, 0, -14) - (0, 0, 1, 0, 8) + 2\,(0, 0, 0, 1, 3)$,
and so $(-1, 1, -1, 2, -5)$ is in the span of the four original vectors.

0026 - 2. Let $S \subseteq \mathbb{R}^4$ be the span of
(1, 1, 1, 5),
(1, 2, 3, 4),
(0, 1, 2, -1),
(2, 2, 2, 10),
(1, 4, 8, 0),
(0, 4, 8, -4).
Extract a basis of S from these six vectors. Your answer should be a subset of the set of these six vectors.

$(0, 1, 2, -1) = (1, 2, 3, 4) - (1, 1, 1, 5)$
$(2, 2, 2, 10) = 2\,(1, 1, 1, 5)$
$(0, 4, 8, -4) = 4\,(0, 1, 2, -1)$
It remains to check that (1, 1, 1, 5), (1, 2, 3, 4), and (1, 4, 8, 0) are linearly independent.
Using row operations on

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 8 & 0 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Therefore, the required basis is (1, 1, 1, 5), (1, 2, 3, 4), and (1, 4, 8, 0).

0026 - 3. Are the vectors
(2, 5, 9, 3),
(2, 4, 6, 8),
(0, 1, 2, -1),
(2, 8, 16, -4)
linearly independent? If not, express one as a linear combination of the others.

They aren' t :
2 (2, 5, 9, 3) - (2, 4, 6, 8) + 2 (0, 1, 2, -1) = (2, 8, 16, -4).

0026 – 4. Find the kernel of M = $\begin{pmatrix} 0 & 1 & -1 & 5 & -7 \\ 1 & 8 & -7 & 38 & -65 \\ 5 & -2 & 2 & 5 & 4 \\ -3 & 6 & -5 & 19 & -34 \end{pmatrix}$.

We use row operations to reduce M to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -11 \\ 0 & 1 & 0 & 0 & -14 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

Therefore, the kernel is the span of the vector (11, 14, −8, −3, 1).

0026 – 5. Find the dimension of the image of

$$M = \begin{pmatrix} 0 & 1 & -1 & 5 & -7 \\ 1 & 8 & -7 & 38 & -65 \\ 5 & -2 & 2 & 5 & 4 \\ -3 & 6 & -5 & 19 & -34 \end{pmatrix}.$$

From 0026 - 4, we have that dim (ker (M)) = 1. Since dim (ker (M)) + dim (im (M)) = 5, dim (im (M)) = 4.

0026 - 6.

a) For each of the following two matrices, compute the dimension of its kernel and the dimension of its image.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -3 \\ -2 & -5 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ -2 & -5 & 3 \end{pmatrix}$$

b) For each of the following two matrices, compute the dimension of its kernel and the dimension of its image.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -3 \\ -2 & -5 & -7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -3 \\ 0 & -2 & -10 \end{pmatrix}$$

a) The hint shows us that the two matrices are related by an elementary column operation so that the dimensions of their kernels are equal and the dimensions of their images are equal.

We row reduce to obtain $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

so that the dimension of each kernels is 0 and the dimension of each image is 3.

b) The two matrices are now related by an elementary row operation,
but we again conclude that the dimensions of their kernels are equal and the dimensions of their images are equal.

We row reduce to obtain $\begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$,

so that the dimension of each kernels is 1 and the dimension of each image is 2.

0026 - 7.

a) Determine which of these two matrices is invertible :

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 1 \\ -2 & -3 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

b) Invert it.

a) We row reduce $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}$ to obtain $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Since the dimension of the kernel is nonzero, this matrix is not invertible.

On the other hand, row reducing $\begin{pmatrix} -1 & -1 & 1 \\ -2 & -3 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ gives the identity so that it is invertible.

b) $\begin{pmatrix} -\frac{9}{2} & 2 & \frac{1}{2} \\ 2 & -1 & 0 \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix}$

0026 - 8. Solve

$x + 3y + 2z = p$

$-2x - 2y - 3z = q$

$3x + 14y + 7z = r$,

where p, q, and r are arbitrary.

We encode the system of equation in the matrix

$$\begin{pmatrix} 1 & 3 & 2 & p \\ -2 & -2 & -3 & q \\ 3 & 14 & 7 & r \end{pmatrix}$$

We then row reduce to obtain the identity in the leftmost three columns

$$\begin{pmatrix} 1 & 0 & 0 & -28p - 7q + 5r \\ 0 & 1 & 0 & -5p - q + r \\ 0 & 0 & 1 & 22p + 5q - 4r \end{pmatrix}$$

Therefore, $x = -28p - 7q + 5r$, $y = -5p - q + r$, $z = 22p + 5q - 4r$.