Financial Mathematics 5001 : Homework 8 (0035)

Due on 7 December 2011

Scot Adams

Solutions

0035–1. Let e1, e2, and e3 be the standard basis of $\mathbb{R}^{1\times 3}$.

a) Let v1 = (2, 4, –2) and v2 = (1, –1, –1). Find a rotation matrix R and scalars c1, c2, such that c1 e1 R = v1 and c2 e2 R = v2.

b) Find a rotation matrix L such that v1 L $\in \mathbb{R}$e1 and v2 L $\in \mathbb{R}$e2.

a) It is clear that since R is a rotation matrix (and therefore length – preserving),

we must take the first row of R to be $\left(\frac{2}{\sqrt{24}}, \frac{4}{\sqrt{24}}, -\frac{2}{\sqrt{24}}\right)$ and the second row to be

$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. To find the third row, we use the fact that rotation matrices are orthonormal. Thus,

if the third row is (x1, x2, x3), then we have the system of three equations :

$2\,x1 + 4\,x2 - 2\,x3 = 0,$

$x1 - x2 - x3 = 0,$ and

$x1^2 + x2^2 + x3^2 = 1.$

The solution that gives R a determinant of one is $x1 = -\frac{1}{\sqrt{2}}$, $x2 = 0$, and $x3 = -\frac{1}{\sqrt{2}}$. Thus,

$$R = \begin{pmatrix} \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \left(\text{and } c1 = \sqrt{24}\,,\ c2 = \sqrt{3}\,\right).$$

b) We take $L = R^{-1}$, since $c1\,e1\,R = v1$ is equivalent to $c1\,e1 = v1\,R^{-1}$, so that $v1\,R^{-1} \in \mathbb{R}e1$ (and likewise for v2, e2).

$$\text{Thus, } L = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

0035–2. Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$. Find rotation matrices $K \in \mathbb{R}^{2\times 2}$ and $L \in \mathbb{R}^{3\times 3}$ such that KML is "diagonal".

$MM^t = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, which has eigenvalues 9 and 4 , with associated eigenvectors (1, 2) and (−2, 1). Normalizing, let $K = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$.

On the other hand, $M^t\, M = \begin{pmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{pmatrix}$, which has eigenvalues 9, 4,

and 0 with associated eigenvectors (5, 2, 4), (0, −2, 1), and (−2, 1, 3). Normalizing, let $L = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & -\frac{2}{3} \\ \frac{2}{3\sqrt{5}} & -\frac{2}{\sqrt{5}} & \frac{1}{3} \\ \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \end{pmatrix}$.

Then the singular value decompostiion theorem implies that KML = D, where $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.
