Financial Mathematics 5001 : Homework 9 (0029 - 0031)

Due on 24 November 2010

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Solutions

0029 - 1. Let $B : \mathbb{R}^5 \times \mathbb{R}^5 \longrightarrow \mathbb{R}$ be the bilinear form defined by

$$[B] = \begin{pmatrix} 3 & 7 & 9 & 0 & 2 \\ 5 & 6 & 8 & 0 & 2 \\ 1 & -2 & -4 & -7 & -8 \\ 4 & -2 & 3 & -4 & 5 \\ 0 & 4 & 6 & 9 & 0 \end{pmatrix}.$$

a. Let v = (1, 2, 1, 0, 0), w = (1, 0, -1, 1, 0). Compute B (v, w).

b. Define $Q : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ by $Q(v) = B(v, v)$. Write out $Q(p, q, r, s, t)$.

c. Find a symmetric matrix $M \in \mathbb{R}^{5\times5}$ such that if S is the SBF defined by $[S] = M$, then $S(v, v) = B(v, v)$.

a. B (v, w) = -14

b. $Q(p, q, r, s, t) =$

$$p(3p + 7q + 9r + 2t) + q(5p + 6q + 8r + 2t) + r(p - 2q - 4r - 7s - 8t) + s(4p - 2q + 3r - 4s + 5t) + t(4q + 6r + 9s) = 3p^2 + 6q^2 - 4r^2 - 4s^2 + 12pq + 10pr + 6qr + 4ps - 2qs - 4rs + 2pt + 6qt - 2rt + 14st$$

c. We can read this matrix off from the coefficients of $Q(p, q, r, s, t)$:

$$M = \begin{pmatrix} 3 & 6 & 5 & 2 & 1 \\ 6 & 6 & 3 & -1 & 3 \\ 5 & 3 & -4 & -2 & -1 \\ 2 & -1 & -2 & -4 & 7 \\ 1 & 3 & -1 & 7 & 0 \end{pmatrix}.$$

0029 - 2. Let $Q: \mathbb{R}^5 \to \mathbb{R}$ be the quatric form defined by

$Q(p,q,r,s,t) = 2p^2 + 2pq + 4q^2 + 4pr - 6qr - 7r^2 - 6ps - 8qs - 18rs - 9s^2 - 6pt + 4qt + 2rt + 6st$.

Let $B: \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ be the polarization of Q. Write out the matrix [B] of B.

$$\begin{pmatrix} 2 & 1 & 2 & -3 & -3 \\ 1 & 4 & -3 & -4 & 2 \\ 2 & -3 & -7 & -9 & 1 \\ -3 & -4 & -9 & -9 & 3 \\ -3 & 2 & 1 & 3 & 0 \end{pmatrix}$$

0029 – 3. Let $Q : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form defined by $Q(x, y) = 2x^2 + 2xy + y^2$. Determine whether Q is positive semidefinite.

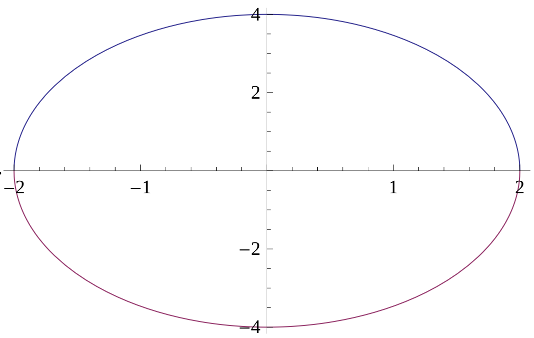
$Q(x, y) = x^2 + (x + y)^2 \geq 0$.

0029 – 4. Let $P : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form defined by $P(x, y) = \frac{x^2}{4} + \frac{y^2}{16}$.

a. Graph $\{(x, y) \mid P(x, y) = 1\}$.

b. Let $v = (2, 8)$. Let B be the polarization of P.

Find a vector $w \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $B(v, w) = 0$.

a. 

b. $[B] = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$. Then $B(v, w) = B((2, 8), (w1, w2)) = \frac{w1}{2} + \frac{w2}{2}$. Thus, $B(v, w) = 0$ if $w = (a, -a)$, with $a \neq 0$.

0030 – 1. Let $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the polarization of the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ defined by $Q(x, y) = 5x^2 + 3y^2$.

a. Compute $B((1, 0), (1, 0))$.

b. Compute $B((1, 0), (0, 1))$.

c. Compute $B((0, 1), (1, 0))$.

d. Compute $B((0, 1), (0, 1))$.

e. Let S be the set of all pairs $(v, w) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $Q(v) \leq 2$ and $Q(w) \leq 8$. Find $\max\{B(v, w) \mid v, w \in S\}$.

a. $[B] = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$. Thus, $B((1, 0), (1, 0)) = 5$.

b. $B((1, 0), (0, 1)) = 0$.

c. $B((0, 1), (1, 0)) = 0$.

d. $B((0, 1), (0, 1)) = 3$.

e. By Cauchy – Schwarz, $\left|B(v, w)\right| \leq \sqrt{Q(v)}\ \sqrt{Q(w)} = 4$.

0030 – 2. Let $R : \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form defined by $R(x, y, z) = 3x^2 + 2y^2 + z^2$.

Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the polarization of R.

Let $D = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, R(x, y, z) \leq 15\right\}$.

Let $v0 = (1, 2, 2)$.

Find $w0 \in D$ such that $B(v0, w0) = \max\{B(v0, w) \mid w \in D\}$.

By Cauchy – Schwarz, $\left|B(v0, w)\right| \leq \sqrt{R(v0)}\sqrt{R(w)} \leq 15$. On the other hand,
$B(v0, w) = B((1, 2, 2), (w1, w2, w3)) = 3\,w1 + 4\,w2 + 2\,w3$. Thus,
we could take w to be, for example, (1, 3, 0) or (1, 2, 2).

0031 – 1. Find a 4×4 rotation matrix whose first column has entries $-\frac{1}{\sqrt{7}}, \sqrt{\frac{2}{7}}, \frac{1}{\sqrt{7}}, \sqrt{\frac{3}{7}}$.

Using Gram – Schmidt, we obtain

$$\begin{pmatrix} -\frac{1}{\sqrt{7}} & \sqrt{\frac{2}{35}} & \frac{1}{2\sqrt{5}} & \frac{\sqrt{3}}{2} \\ \sqrt{\frac{2}{7}} & \sqrt{\frac{5}{7}} & 0 & 0 \\ \frac{1}{\sqrt{7}} & -\sqrt{\frac{2}{35}} & \frac{2}{\sqrt{5}} & 0 \\ \sqrt{\frac{3}{7}} & -\sqrt{\frac{6}{35}} & -\frac{\sqrt{\frac{3}{5}}}{2} & \frac{1}{2} \end{pmatrix}$$

0031 – 2. Find a 2×2 rotation matrix M such that $L_M\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

These vectors lie on the unit circle. The first corresponds to the angle $\frac{2\pi}{3}$,

the second to $-\frac{\pi}{4}$. Thus, the angle of rotation is $-$

$\frac{11\pi}{12}$. This corresponds to the rotation matrix $\begin{pmatrix} \mathrm{Cos}\left[-\frac{11\pi}{12}\right] & -\mathrm{Sin}\left[-\frac{11\pi}{12}\right] \\ \mathrm{Sin}\left[-\frac{11\pi}{12}\right] & \mathrm{Cos}\left[-\frac{11\pi}{12}\right] \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} \\ -\frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$.

0031 - 3.

a. Compute $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b. Show that the rows of R are orthonormal.

c. Compute R^{-1}.

a. $R = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3\sqrt{3}}{10} & \frac{2\sqrt{3}}{5} & -\frac{1}{2} \\ \frac{3}{10} & \frac{2}{5} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

b. R is a rotation matrix because it is the product of two rotation matrices, and so has orthonomal rows.

c. $R^{-1} = R^T = \begin{pmatrix} \frac{4}{5} & \frac{3\sqrt{3}}{10} & \frac{3}{10} \\ -\frac{3}{5} & \frac{2\sqrt{3}}{5} & \frac{2}{5} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.