Financial Mathematics 5002 : Unassigned Homework (0036, 0037)

Not due on 14 December 2011

Scot Adams

Solutions

0036–1. Let $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & -4 \\ 3 & -2 & 0 \end{pmatrix}$. Let $d = \det A$. Let B be the cofactor matrix of A. Let I be the 3×3 identity matrix.

a) Find B.

b) Find B^t.

c) Find AB^t.

d) Find $B^t A$.

e) Find dI.

a) $B = \begin{pmatrix} -8 & -12 & -10 \\ 4 & 6 & 8 \\ -4 & 0 & -2 \end{pmatrix}$.

b) $B^t = \begin{pmatrix} -8 & 4 & -4 \\ -12 & 6 & 0 \\ -10 & 8 & -2 \end{pmatrix}$.

c), d), e) $AB^t = B^t A = dI = \begin{pmatrix} -12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -12 \end{pmatrix}$.

0036–2. Let $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & -4 \\ 3 & -2 & 0 \end{pmatrix}$. Le I be the 3x3 identity matrix. Let $f(\lambda)$ be the characteristic polynomial of A. Then $f(\lambda)$=det$(A-\lambda I) = -\lambda^3 + p\lambda^2 + q\lambda + r$ for some $p,q,r \in \mathbb{Z}$.

a. Find p, q, r.

b. Find A^2 and A^3

c. Find $-A^3 + pA^2 + qA + rI$.

a. $\det(A) = -12 + 4\lambda + 3\lambda^2 - \lambda^3$, so that $p = 3$, $q = 4$, and $r = -12$.

b. $A^2 = \begin{pmatrix} -1 & 10 & -10 \\ -6 & 16 & -12 \\ -1 & 2 & 2 \end{pmatrix}$, $A^3 = \begin{pmatrix} -11 & 38 & -38 \\ -10 & 44 & -52 \\ 9 & -2 & -6 \end{pmatrix}$.

c. By Cayley's Theorem, we know that A satisfies its own characteristic polynomial. But we can still do the computation :

$-A^3 + pA^2 + qA + rI =$

$$-A^3 + 3A^2 + 4A + -12I = -\begin{pmatrix} -11 & 38 & -38 \\ -10 & 44 & -52 \\ 9 & -2 & -6 \end{pmatrix} + 3\begin{pmatrix} -1 & 10 & -10 \\ -6 & 16 & -12 \\ -1 & 2 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & -4 \\ 3 & -2 & 0 \end{pmatrix} - 12\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

0037 – 1. Let $f(x, y) = e^{2x-y} \sin(x+y)$.

a. Compute $\frac{\partial}{\partial x}[f(x, y)]$.

b. Compute $\left[\frac{\partial}{\partial x}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4}$.

c. Compute $\frac{\partial}{\partial y}[f(x, y)]$.

d. Compute $\left[\frac{\partial}{\partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4}$.

e. Compute the gradient of $f(x, y)$ with respect to (x, y).

f. Evaluate the gradient of $f(x, y)$ with respect to (x, y) at the point $(x, y) = (\pi/4, \pi/4)$.

g. Find the 1 –jet of $f(x, y)$ with respect to (x, y) at the point $(x, y) = (\pi/4, \pi/4)$.

h. Find the 1 st order Maclaurin approximation of $f(x + \pi/4, y + \pi/4)$.

i. Compute $\frac{\partial^2}{\partial x^2}[f(x, y)]$.

j. Compute $\left[\frac{\partial^2}{\partial x^2}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4}$.

k. Compute $\frac{\partial^2}{\partial x\,\partial y}[f(x, y)]$.

l. Compute $\left[\frac{\partial^2}{\partial x\,\partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4}$.

m. Compute $\frac{\partial^2}{\partial y^2}[f(x, y)]$.

n. Compute $\left[\frac{\partial^2}{\partial y^2}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4}$.

o. Compute the 2 x2 Hessian of $f(x, y)$ with respect to (x, y).

p. Evaluate the 2 x2 Hessian of $f(x, y)$ with respect to (x, y) at the point $(x, y) = (\pi/4, \pi/4)$.

q. Find the 2 –jet of $f(x, y)$ with respect to (x, y) at the point $(x, y) = (\pi/4, \pi/4)$. (Use lexicographic ordering.)

r. Find the 2 nd order Maclaurin approximation of f $(x + \pi/4, y + \pi/4)$.

a. $\frac{\partial}{\partial x}[f(x, y)] = e^{2x-y}\text{Cos}[x+y] + 2e^{2x-y}\text{Sin}[x+y]$.

b. $\left[\frac{\partial}{\partial x}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = 2e^{\pi/4}$.

c. $\frac{\partial}{\partial y}[f(x, y)] = e^{2x-y}\text{Cos}[x+y] - e^{2x-y}\text{Sin}[x+y]$.

d. $\left[\frac{\partial}{\partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = -e^{\pi/4}$.

e. $\nabla f(x, y) = \left(e^{2x-y}\text{Cos}[x+y] + 2e^{2x-y}\text{Sin}[x+y],\ e^{2x-y}\text{Cos}[x+y] - e^{2x-y}\text{Sin}[x+y]\right)$.

f. $\nabla f(\pi/4, \pi/4) = \left(2e^{\pi/4}, -e^{\pi/4}\right)$.

g. $J^1 f(\pi/4, \pi/4) = \left(e^{\pi/4}, 2e^{\pi/4}, -e^{\pi/4}\right)$.

h. $f(\pi/4, \pi/4) + x\left[\frac{\partial}{\partial x}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} + y\left[\frac{\partial}{\partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = e^{\pi/4} + 2e^{\pi/4}x - e^{\pi/4}y$.

i. $\frac{\partial^2}{\partial x^2}[f(x, y)] = 4e^{2x-y}\text{Cos}[x+y] + 3e^{2x-y}\text{Sin}[x+y]$.

j. $\left[\frac{\partial^2}{\partial x^2}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = 3e^{\pi/4}$.

k. $\frac{\partial^2}{\partial x\,\partial y}[f(x, y)] = e^{2x-y}\text{Cos}[x+y] - 3e^{2x-y}\text{Sin}[x+y]$.

l. $\left[\frac{\partial^2}{\partial x\,\partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = -3e^{\pi/4}$.

m. $\frac{\partial^2}{\partial y^2}[f(x, y)] = -2e^{2x-y}\text{Cos}[x+y]$.

n. $\left[\frac{\partial^2}{\partial y^2}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = 0$.

o. $Hf = \begin{pmatrix} 4e^{2x-y}\text{Cos}[x+y] + 3e^{2x-y}\text{Sin}[x+y] & e^{2x-y}\text{Cos}[x+y] - 3e^{2x-y}\text{Sin}[x+y] \\ e^{2x-y}\text{Cos}[x+y] - 3e^{2x-y}\text{Sin}[x+y] & -2e^{2x-y}\text{Cos}[x+y] \end{pmatrix}$

p. $Hf(\pi/4, \pi/4) = \begin{pmatrix} 3e^{\pi/4} & -3e^{\pi/4} \\ -3e^{\pi/4} & 0 \end{pmatrix}$.

q. $J^2 f(\pi/4, \pi/4) = \left(e^{\pi/4}, 2e^{\pi/4}, -e^{\pi/4}, 3e^{\pi/4}, -3e^{\pi/4}, 0\right)$.

r. $e^{\pi/4} + 2 e^{\pi/4} x - e^{\pi/4} y + \left(3 e^{\pi/4}\right) \frac{x^2}{2} + \left(-3 e^{\pi/4}\right) x\, y.$

0037-2. How many terms appear in the 6 th order Maclaurin approximation of a function of 9 variables?

This is equivalent to asking for the number of monomials of degree less than or equal to 6 on 9 variables, which is just $\binom{6+9}{9} = 5005$.

0037-3. Find the second - order Maclaurin approximation, q (x, y), (with respect to (x, y)) to the expression $g(x, y) = \tan(e^{2x+y}) + 2xy^2 - 4\cos(x) + 8x$.

We compute :

$[g]_{x\to 0,\, y\to 0} = -4 + \mathrm{Tan}[1]$.

$\frac{\partial g}{\partial x} = 8 + 2y^2 + 2 e^{2x+y} \mathrm{Sec}[e^{2x+y}]^2 + 4\,\mathrm{Sin}[x]$, $\left[\frac{\partial g}{\partial x}\right]_{x\to 0,\, y\to 0} = 8 + 2\,\mathrm{Sec}[1]^2$.

$\frac{\partial g}{\partial y} = 4xy + e^{2x+y} \mathrm{Sec}[e^{2x+y}]^2$, $\left[\frac{\partial g}{\partial y}\right]_{x\to 0,\, y\to 0} = \mathrm{Sec}[1]^2$.

$\frac{\partial^2 g}{\partial x^2} = 4\,\mathrm{Cos}[x] + 4 e^{2x+y} \mathrm{Sec}[e^{2x+y}]^2 + 8 e^{4x+2y} \mathrm{Sec}[e^{2x+y}]^2 \mathrm{Tan}[e^{2x+y}]$, $\left[\frac{\partial^2 g}{\partial x^2}\right]_{x\to 0,\, y\to 0} = 4 + 4\,\mathrm{Sec}[1]^2 + 8\,\mathrm{Sec}[1]^2\,\mathrm{Tan}[1]$.

$\frac{\partial^2 g}{\partial x \partial y} = 4y + 2 e^{2x+y} \mathrm{Sec}[e^{2x+y}]^2 + 4 e^{4x+2y} \mathrm{Sec}[e^{2x+y}]^2 \mathrm{Tan}[e^{2x+y}]$, $\left[\frac{\partial^2 g}{\partial x \partial y}\right]_{x\to 0,\, y\to 0} = 2\,\mathrm{Sec}[1]^2 + 4\,\mathrm{Sec}[1]^2\,\mathrm{Tan}[1]$.

$\frac{\partial^2 g}{\partial y^2} = 4x + e^{2x+y} \mathrm{Sec}[e^{2x+y}]^2 + 2 e^{4x+2y} \mathrm{Sec}[e^{2x+y}]^2 \mathrm{Tan}[e^{2x+y}]$, $\left[\frac{\partial^2 g}{\partial y^2}\right]_{x\to 0,\, y\to 0} = \mathrm{Sec}[1]^2 + 2\,\mathrm{Sec}[1]^2\,\mathrm{Tan}[1]$.

Therefore, the second order Maclaurin approximation is $(-4 + \mathrm{Tan}[1]) + (8 + 2\,\mathrm{Sec}[1]^2)\, x + (\mathrm{Sec}[1]^2)\, y +$
$(4 + 4\,\mathrm{Sec}[1]^2 + 8\,\mathrm{Sec}[1]^2\,\mathrm{Tan}[1]) \frac{x^2}{2} + (2\,\mathrm{Sec}[1]^2 + 4\,\mathrm{Sec}[1]^2\,\mathrm{Tan}[1])\, x\, y + (\mathrm{Sec}[1]^2 + 2\,\mathrm{Sec}[1]^2\,\mathrm{Tan}[1]) \frac{y^2}{2}$.

0037-4. Let f (x, y) =
$\sqrt{7^2 - x^2 - 2y^2}$. (Note : the graph of $z = f(x, y)$ is the upper half of the ellipsoid $x^2 + 2y^2 + z^2 = 7^2$. The lower half would b ethe graph of $z = -f(x, y)$. The north pole appears on the graph of $z = f(x, y)$ over $(x, y) = (0, 0)$ at $(0, 0, 7)$.)

a. Compute $(\nabla f)(0, 0) \in \mathbb{R}^2$.

b. Compute (f ') (0, 0) $\in \mathbb{R}^{1\times 2}$.

c. Compute (Hf) (0, 0) $\in \mathbb{R}^{2\times 2}$.

d. Compute (f '') (0, 0) $\in \mathbb{R}^{2\times 2}$.

e. Compute $Q_{f''(0,0)}(x, y)$.

f. Show $Q_{f''(0,0)}(x, y)$ is negative definite.

a. $\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{49 - x^2 - 2y^2}}$, $\frac{\partial f}{\partial x} = -\frac{2y}{\sqrt{49 - x^2 - 2y^2}}$, so $(\nabla f)(0, 0) = (0, 0)$.

b. Again, this is $\begin{pmatrix} 0 & 0 \end{pmatrix}$.

c. We compute :

$$\frac{\partial^2 g}{\partial x^2} = -\frac{x^2}{(49 - x^2 - 2y^2)^{3/2}} - \frac{1}{\sqrt{49 - x^2 - 2y^2}},\ \left[\frac{\partial^2 g}{\partial x^2}\right]_{x\to 0,\, y\to 0} = -\frac{1}{7}.$$

$$\frac{\partial^2 g}{\partial x\, \partial y} = -\frac{2xy}{(49 - x^2 - 2y^2)^{3/2}},\ \left[\frac{\partial^2 g}{\partial x\, \partial y}\right]_{x\to 0,\, y\to 0} = 0.$$

$$\frac{\partial^2 g}{\partial y^2} = -\frac{4y^2}{(49 - x^2 - 2y^2)^{3/2}} - \frac{2}{\sqrt{49 - x^2 - 2y^2}},\ \left[\frac{\partial^2 g}{\partial y^2}\right]_{x\to 0,\, y\to 0} = -\frac{2}{7}.$$

Therefore, $(Hf)(0, 0) = \begin{pmatrix} -\frac{1}{7} & 0 \\ 0 & -\frac{2}{7} \end{pmatrix}$.

d. Again, this is $\begin{pmatrix} -\frac{1}{7} & 0 \\ 0 & -\frac{2}{7} \end{pmatrix}$.

e. We compute the bilinear form by $Q_{f''(0,0)}(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -\frac{1}{7} & 0 \\ 0 & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{7}(x^2 + 2y^2)$.

f. $(x^2 + 2y^2)$ is a sum of a square and twice a square, and is therefore non − negative. Therefore, $Q_{f''(0,0)}(x, y) = -\frac{1}{7}(x^2 + 2y^2)$ is non − positive, or negative definite.

0037−5. Let $Q(x, y) = 3x^2 - 4xy + 3y^2$. Let $f(x, y) = \sqrt{1 - Q(x, y)}$. Let $M = (Hf)(0, 0) \in \mathbb{R}^{2\times 2}$.

a. Show that (0, 0) is a critical point for f—that is, show that $(\nabla f)(0, 0) = (0, 0)$.

b. Compute M.

c. Find the characterisitc polynomial of M.

d. Find the eigenvalues of M.

e. For each eigenvalue of M, find a basis of the corresponding eigenspace.

f. Find a 2x2 rotation matrix R such that $R^{-1} MR$ is diagonal.

a. $\frac{\partial f}{\partial x} = \frac{-6x+4y}{2\sqrt{1-3x^2+4xy-3y^2}}$, $\frac{\partial f}{\partial y} = \frac{4x-6y}{2\sqrt{1-3x^2+4xy-3y^2}}$, so $(\nabla f)(0,0) = (0,0)$.

b. We compute :

$$\frac{\partial^2 f}{\partial x^2} = -\frac{(-6x+4y)^2}{4(1-3x^2+4xy-3y^2)^{3/2}} - \frac{3}{\sqrt{1-3x^2+4xy-3y^2}}, \left[\frac{\partial^2 f}{\partial x^2}\right]_{x\to 0,\, y\to 0} = -3.$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{(4x-6y)(-6x+4y)}{4(1-3x^2+4xy-3y^2)^{3/2}} + \frac{2}{\sqrt{1-3x^2+4xy-3y^2}}, \left[\frac{\partial^2 g}{\partial x \partial y}\right]_{x\to 0,\, y\to 0} = 2.$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{(4x-6y)^2}{4(1-3x^2+4xy-3y^2)^{3/2}} - \frac{3}{\sqrt{1-3x^2+4xy-3y^2}}, \left[\frac{\partial^2 f}{\partial y^2}\right]_{x\to 0,\, y\to 0} = -3.$$

Therefore, $(Hf)(0,0) = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$.

c. $\det\begin{pmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{pmatrix} = 5 + 6\lambda + \lambda^2$

d. Solving $5 + 6\lambda + \lambda^2 = 0$ gives $\lambda = -5$ or $\lambda = -1$, which are the eigenvalues.

e. A basis for the eigenspace corresponding to the eigenvalue -5 is given by the vector $(-1, 1)$. The eigenspace corresponding to -1 is spanned by $(1, 1)$.

f. Normalizing the eigenvectors and arranging them as the columns of the matrix R, we obtain $R = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.