Financial Mathematics 5002 : Homework 1 (0037)

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Solutions

0037 – 1. Let $f(x, y) = e^{x^2 - y^2} \sin(x + y)$.

a. Compute $\frac{\partial}{\partial x}[f(x, y)]$.

b. Compute $\left[\frac{\partial}{\partial x}[f(x, y)]\right]_{x \to \pi/4,\, y \to \pi/4}$.

c. Compute $\frac{\partial}{\partial y}[f(x, y)]$.

d. Compute $\left[\frac{\partial}{\partial y}[f(x, y)]\right]_{x \to \pi/4,\, y \to \pi/4}$.

e. Compute the gradient of $f(x, y)$ with respect to (x, y).

f. Evaluate the gradient of $f(x, y)$ with respect to (x, y) at the point $(x, y) = (\pi/4, \pi/4)$.

g. Find the 1 - jet of $f(x, y)$ with respect to (x, y) at the point $(x, y) = (\pi/4, \pi/4)$.

h. Find the 1 st order Maclaurin approximation of $f(x + \pi/4, y + \pi/4)$.

i. Compute $\frac{\partial^2}{\partial x^2}[f(x, y)]$.

j. Compute $\left[\frac{\partial^2}{\partial x^2}[f(x, y)]\right]_{x \to \pi/4,\, y \to \pi/4}$.

k. Compute $\frac{\partial^2}{\partial x \partial y}[f(x, y)]$.

l. Compute $\left[\frac{\partial^2}{\partial x \partial y}[f(x, y)]\right]_{x \to \pi/4,\, y \to \pi/4}$.

m. Compute $\frac{\partial^2}{\partial y^2}[f(x, y)]$.

n. Compute $\left[\frac{\partial^2}{\partial y^2}[f(x, y)]\right]_{x \to \pi/4,\, y \to \pi/4}$.

o. Compute the 2 x2 Hessian of f (x, y) with respect to (x, y).

p. Evaluate the 2 x2 Hessian of f (x, y) with respect to (x, y) at the point (x, y) = (π/4, π/4).

q. Find the 2 - jet of f (x, y) with respect to (x, y) at the point (x, y) = (π/4, π/4). (Use lexicographic ordering.)

r. Find the 2 nd order Maclaurin approximation of f (x + π/4, y + π/4).

a. $\frac{\partial}{\partial x}[f(x, y)] = e^{x^2-y^2} \text{Cos}[x+y] + 2 e^{x^2-y^2} x \text{Sin}[x+y]$.

b. $\left[\frac{\partial}{\partial x}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = \frac{\pi}{2}$.

c. $\frac{\partial}{\partial y}[f(x, y)] = e^{x^2-y^2} \text{Cos}[x+y] - 2 e^{x^2-y^2} y \text{Sin}[x+y]$.

d. $\left[\frac{\partial}{\partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = -\frac{\pi}{2}$.

e. $\nabla f(x, y) = \left(e^{x^2-y^2} \text{Cos}[x+y] + 2 e^{x^2-y^2} x \text{Sin}[x+y],\ e^{x^2-y^2} \text{Cos}[x+y] - 2 e^{x^2-y^2} y \text{Sin}[x+y]\right)$.

f. $\nabla f(\pi/4, \pi/4) = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.

g. $J^1 f(\pi/4, \pi/4) = \left(1, \frac{\pi}{2}, -\frac{\pi}{2}\right)$.

h. $f(\pi/4, \pi/4) + x \left[\frac{\partial}{\partial x}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} + y\left[\frac{\partial}{\partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = 1 + \frac{\pi x}{2} - \frac{\pi y}{2}$.

i. $\frac{\partial^2}{\partial x^2}[f(x, y)] = 4 e^{x^2-y^2} x \text{Cos}[x+y] + e^{x^2-y^2} \text{Sin}[x+y] + 4 e^{x^2-y^2} x^2 \text{Sin}[x+y]$.

j. $\left[\frac{\partial^2}{\partial x^2}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = 1 + \frac{\pi^2}{4}$.

k. $\frac{\partial^2}{\partial x \partial y}[f(x, y)] = 2 e^{x^2-y^2} x \text{Cos}[x+y] - 2 e^{x^2-y^2} y \text{Cos}[x+y] - e^{x^2-y^2} \text{Sin}[x+y] - 4 e^{x^2-y^2} x y \text{Sin}[x+y]$.

l. $\left[\frac{\partial^2}{\partial x \partial y}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = -1 - \frac{\pi^2}{4}$.

m. $\frac{\partial^2}{\partial y^2}[f(x, y)] = -4 e^{x^2-y^2} y \text{Cos}[x+y] - 3 e^{x^2-y^2} \text{Sin}[x+y] + 4 e^{x^2-y^2} y^2 \text{Sin}[x+y]$.

n. $\left[\frac{\partial^2}{\partial y^2}[f(x, y)]\right]_{x\to\pi/4,\, y\to\pi/4} = -3 + \frac{\pi^2}{4}.$

o. $Hf = \begin{pmatrix} e^{x^2-y^2}(\text{Sin}[x+y] + 4x(\text{Cos}[x+y] + x\,\text{Sin}[x+y])) & e^{x^2-y^2}(2(x-y)\text{Cos}[x+y] - (1+4xy)\text{Sin}[x+y]) \\ e^{x^2-y^2}(2(x-y)\text{Cos}[x+y] - (1+4xy)\text{Sin}[x+y]) & e^{x^2-y^2}\left(-4y\,\text{Cos}[x+y] + \left(-3+4y^2\right)\text{Sin}[x+y]\right) \end{pmatrix}$

p. $Hf(\pi/4, \pi/4) = \begin{pmatrix} 1+\frac{\pi^2}{4} & -1-\frac{\pi^2}{4} \\ -1-\frac{\pi^2}{4} & -3+\frac{\pi^2}{4} \end{pmatrix}.$

q. $J^2 f(\pi/4, \pi/4) = \left(1, \frac{\pi}{2}, -\frac{\pi}{2}, 1+\frac{\pi^2}{4}, -1-\frac{\pi^2}{4}, -3+\frac{\pi^2}{4}\right).$

r. $1 + \frac{\pi x}{2} - \frac{\pi y}{2} + \left(1+\frac{\pi^2}{4}\right)\frac{x^2}{2} + \left(-1-\frac{\pi^2}{4}\right)x\,y + \left(-3+\frac{\pi^2}{4}\right)\frac{y^2}{2}.$

0037 - 2. How many terms appear in the 9 th order Maclaurin approximation of a function of 6 variables?

This is equivalent to asking for the number of monomials of degree less than or equal to 9 on 6 variables, which is just $\binom{9+6}{6} = 5005.$

0037 – 3. Find the second – order Maclaurin approximation, $q(x, y)$, (with respect to (x, y)) to the expression $g(x, y) = \sin\left(e^{x+3y}\right) + 2xy^3 - 5\tan(y) - 2.$

We compute :

$[g]_{x\to0,\, y\to0} = \text{Sin}[1] - 2.$

$\frac{\partial g}{\partial x} = 2y^3 + e^{x+3y}\,\text{Cos}\left[e^{x+3y}\right],\ \left[\frac{\partial g}{\partial x}\right]_{x\to0,\, y\to0} = \text{Cos}[1].$

$\frac{\partial g}{\partial y} = 6xy^2 + 3e^{x+3y}\,\text{Cos}\left[e^{x+3y}\right] - 5\,\text{Sec}[y]^2,\ \left[\frac{\partial g}{\partial y}\right]_{x\to0,\, y\to0} = -5 + 3\,\text{Cos}[1].$

$\frac{\partial^2 g}{\partial x^2} = e^{x+3y}\,\text{Cos}\left[e^{x+3y}\right] - e^{2x+6y}\,\text{Sin}\left[e^{x+3y}\right],\ \left[\frac{\partial^2 g}{\partial x^2}\right]_{x\to0,\, y\to0} = \text{Cos}[1] - \text{Sin}[1].$

$\frac{\partial^2 g}{\partial x\,\partial y} = 6y^2 + 3e^{x+3y}\,\text{Cos}\left[e^{x+3y}\right] - 3e^{2x+6y}\,\text{Sin}\left[e^{x+3y}\right],\ \left[\frac{\partial^2 g}{\partial x\,\partial y}\right]_{x\to0,\, y\to0} = 3\,\text{Cos}[1] - 3\,\text{Sin}[1].$

$$\frac{\partial^2 g}{\partial y^2} = 12\,x\,y + 9\,e^{x+3\,y}\,\text{Cos}\left[e^{x+3\,y}\right] - 9\,e^{2\,x+6\,y}\,\text{Sin}\left[e^{x+3\,y}\right] - 10\,\text{Sec}[y]^2\,\text{Tan}[y],\quad \left[\frac{\partial^2 g}{\partial y^2}\right]_{x\to 0,\,y\to 0} = 9\,\text{Cos}[1] - 9\,\text{Sin}[1].$$

Therefore, the second order Maclaurin approximation is $\text{Sin}[1] - 2 + \text{Cos}[1]\,x +$

$$(-5 + 3\,\text{Cos}[1])\,y + (\text{Cos}[1] - \text{Sin}[1])\,\frac{x^2}{2} + (3\,\text{Cos}[1] - 3\,\text{Sin}[1])\,x\,y + (9\,\text{Cos}[1] - 9\,\text{Sin}[1])\,\frac{y^2}{2}.$$

0037 – 4. Let $f(x, y) = \sqrt{9^2 - x^2 - y^2}$. (Note : the graph of $z =$ $f(x, y)$ is the upper half of the sphere $x^2 + y^2 + z^2 = 9^2$. The lower half would b ethe graph of $z = -f(x, y)$. The north pole appears on the graph of $z = f(x, y)$ over $(x, y) = (0, 0)$ at $(0, 0, 1)$.)

a. Compute $(\nabla f)(0, 0) \in \mathbb{R}^2$.

b. Compute $(f')(0, 0) \in \mathbb{R}^{1\,x2}$.

c. Compute $(Hf)(0, 0) \in \mathbb{R}^{2\,x2}$.

d. Compute $(f'')(0, 0) \in \mathbb{R}^{2\,x2}$.

e. Compute $Q_{f''(0,0)}(x, y)$.

f. Show $Q_{f''(0,0)}(x, y)$ is negative definite.

a. $\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{81 - x^2 - y^2}},\quad \frac{\partial f}{\partial x} = -\frac{y}{\sqrt{81 - x^2 - y^2}},$ so $(\nabla f)(0, 0) = (0, 0)$.

b. Again, this is $(\,0 \quad 0\,)$.

c. We compute :

$$\frac{\partial^2 g}{\partial x^2} = -\frac{x^2}{(81 - x^2 - y^2)^{3/2}} - \frac{1}{\sqrt{81 - x^2 - y^2}},\quad \left[\frac{\partial^2 g}{\partial x^2}\right]_{x\to 0,\,y\to 0} = -\frac{1}{9}.$$

$$\frac{\partial^2 g}{\partial x\,\partial y} = -\frac{x\,y}{(81 - x^2 - y^2)^{3/2}},\quad \left[\frac{\partial^2 g}{\partial x\,\partial y}\right]_{x\to 0,\,y\to 0} = 0.$$

$$\frac{\partial^2 g}{\partial y^2} = -\frac{y^2}{(81 - x^2 - y^2)^{3/2}} - \frac{1}{\sqrt{81 - x^2 - y^2}},\quad \left[\frac{\partial^2 g}{\partial y^2}\right]_{x\to 0,\,y\to 0} = -\frac{1}{9}.$$

Therefore, $(Hf)(0, 0) = \begin{pmatrix} -\frac{1}{9} & 0 \\ 0 & -\frac{1}{9} \end{pmatrix}$.

d. Again, this is $\begin{pmatrix} -\frac{1}{9} & 0 \\ 0 & -\frac{1}{9} \end{pmatrix}$.

e. We compute the bilinear form by $Q_{f''(0,0)}(x, y) = (x \;\; y)\begin{pmatrix} -\frac{1}{9} & 0 \\ 0 & -\frac{1}{9} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{9}(x^2 + y^2)$.

f. $(x^2 + y^2)$ is a sum of two squares, and is therefore non – negative. Therefore,

$Q_{f''(0,0)}(x, y) = -\frac{1}{9}(x^2 + y^2)$ is non – positive, or negative definite.

0037 – 5. Let $Q(x, y) = \frac{26}{25}x^2 - \frac{86}{25}xy - \frac{1}{25}y^2$. Let $f(x, y) = \sqrt{1 - Q(x, y)}$. Let $M = (Hf)(0, 0) \in \mathbb{R}^{2\,x2}$.

a. Show that (0, 0) is a critical point for f- - that is, show that $(\nabla f)(0, 0) = (0, 0)$.

b. Compute M.

c. Find the characterisitc polynomial of M.

d. Find the eigenvalues of M.

e. For each eigenvalue of M, find a basis of the corresponding eigenspace.

f. Find a 2 x2 rotation matrix R such that $R^{-1}MR$ is diagonal.

a. $\frac{\partial f}{\partial x} = \frac{-\frac{52x}{25} + \frac{86y}{25}}{2\sqrt{1 - \frac{26x^2}{25} + \frac{86xy}{25} + \frac{y^2}{25}}}$, $\frac{\partial f}{\partial y} = \frac{\frac{86x}{25} + \frac{2y}{25}}{2\sqrt{1 - \frac{26x^2}{25} + \frac{86xy}{25} + \frac{y^2}{25}}}$, so $(\nabla f)(0, 0) = (0, 0)$.

b. We compute :

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\left(-\frac{52x}{25} + \frac{86y}{25}\right)^2}{4\left(1 - \frac{26x^2}{25} + \frac{86xy}{25} + \frac{y^2}{25}\right)^{3/2}} - \frac{26}{25\sqrt{1 - \frac{26x^2}{25} + \frac{86xy}{25} + \frac{y^2}{25}}}, \left[\frac{\partial^2 f}{\partial x^2}\right]_{x\to 0,\, y\to 0} = -\frac{26}{25}.$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -\frac{\left(\frac{86x}{25} + \frac{2y}{25}\right)\left(-\frac{52x}{25} + \frac{86y}{25}\right)}{4\left(1 - \frac{26x^2}{25} + \frac{86xy}{25} + \frac{y^2}{25}\right)^{3/2}} + \frac{43}{25\sqrt{1 - \frac{26x^2}{25} + \frac{86xy}{25} + \frac{y^2}{25}}}, \quad \left[\frac{\partial^2 g}{\partial x\,\partial y}\right]_{x\to 0,\, y\to 0} = \frac{43}{25}.$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{\left(\frac{86\,x}{25} + \frac{2\,y}{25}\right)^2}{4\left(1 - \frac{26\,x^2}{25} + \frac{86\,x\,y}{25} + \frac{y^2}{25}\right)^{3/2}} + \frac{1}{25\sqrt{1 - \frac{26\,x^2}{25} + \frac{86\,x\,y}{25} + \frac{y^2}{25}}},\ \left[\frac{\partial^2 f}{\partial y^2}\right]_{x\to 0,\, y\to 0} = \frac{1}{25}.$$

Therefore, $(\mathrm{Hf})(0,\ 0) = \begin{pmatrix} -\frac{26}{25} & \frac{43}{25} \\ \frac{43}{25} & \frac{1}{25} \end{pmatrix}$.

c. $\det\begin{pmatrix} -\frac{26}{25} - \lambda & \frac{43}{25} \\ \frac{43}{25} & \frac{1}{25} - \lambda \end{pmatrix} = -3 + \lambda + \lambda^2$

d. Solving $-3 + \lambda + \lambda^2 = 0$ gives $\lambda = \frac{1}{2}\left(-1 - \sqrt{13}\right)$ or $\lambda = \frac{1}{2}\left(-1 + \sqrt{13}\right)$, which are the eigenvalues.

e. A basis for the eigenspace corresponding to the eigenvalue $\frac{1}{2}\left(-1 - \sqrt{13}\right)$ is given by the vector $\left(-\frac{1}{43} + \frac{25}{86}\left(-1 - \sqrt{13}\right),\ 1\right)$. The eigenspace corresponding to $\frac{1}{2}\left(-1 + \sqrt{13}\right)$ is spanned by $\left(-\frac{1}{43} + \frac{25}{86}\left(-1 + \sqrt{13}\right),\ 1\right)$.

f. Normalizing the eigenvectors and arranging them as the columns of the matrix R, we obtain

$$R = \begin{pmatrix} \frac{-\frac{1}{43}+\frac{25}{86}\left(-1+\sqrt{13}\right)}{\sqrt{1+\left(-\frac{1}{43}+\frac{25}{86}\left(-1+\sqrt{13}\right)\right)^2}} & \frac{-\frac{1}{43}+\frac{25}{86}\left(-1-\sqrt{13}\right)}{\sqrt{1+\left(-\frac{1}{43}+\frac{25}{86}\left(-1-\sqrt{13}\right)\right)^2}} \\ \frac{1}{\sqrt{1+\left(-\frac{1}{43}+\frac{25}{86}\left(-1+\sqrt{13}\right)\right)^2}} & \frac{1}{\sqrt{1+\left(-\frac{1}{43}+\frac{25}{86}\left(-1-\sqrt{13}\right)\right)^2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{5}\sqrt{\frac{25}{2} - \frac{27}{2\sqrt{13}}} & -\sqrt{\frac{1}{2} + \frac{27}{50\sqrt{13}}} \\ \sqrt{\frac{1}{2} + \frac{27}{50\sqrt{13}}} & \frac{1}{5}\sqrt{\frac{25}{2} - \frac{27}{2\sqrt{13}}} \end{pmatrix}.$$