# Financial Mathematics 5002 : Homework 11 (0060 - 0062)

Due on 27 April 2011

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Solutions

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0060 - 1. Let X represent the price, three months from now, of some financial asset. Assume that the expected annual return is 4 %. (That is, if you invest \$1 in the asset today, then its expected value, one year from today, is \$1 .04.)
Assume that 1.04 is the exponential NOT of the drift but rather of the "augmented" drift. Assume that the annualized volatility is 0.35. (In the following problems, use 3 months = 0.25 years.)

a. Find the annual drift.

b. Find the 3 - month volatility.

c. Find the 3 - month drift.

a. Let x be the annual drift. Then $E[S_1] = e^{x+1/2\,(0.35)^2} = 1.04$, so that $x = -0.0220293$.

b. The 3 – month volatility is $\dfrac{0.35}{\sqrt{4}} = 0.175$.

c. The 3 – month drift is $\dfrac{-0.0220293}{4} = -0.00550733$.

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0060 - 2. We analyze a particular stock over a time interval that starts today, and extends 100 days into the future. Assume that the current price is \$5 per share. Assume that the annualized drift is 5 %, that the annualized volatility is 35 %. Use the 75 - 25 400 - subperiod CRR model.

a. Find the 6 hour uptick and downtick factors that calibrate to the above data.

b. Write a summation expression with binomial coefficients for the expected price of the stock at the end of the 100 days.

a. We compute that

$\mu_* = \dfrac{0.05}{4 \times 365} \approx 0.000034246575$, and

$\sigma_* = \dfrac{0.35}{\sqrt{4 \times 365}} \approx 0.0091599186$, so that

$(0.75)\,u + (0.25)\,d = \mu_* \approx 0.000034246575$, and

$\sqrt{(0.75)\,(0.25)}\;\;(u - d) = \sigma_* \approx 0.0091599186.$

Solving, we conclude that
$u \approx 0.0053227230$ and $d \approx -0.015831183$.

b. Using u and d as above, we have the expression for the expected price of the stock

$$5 \sum_{i=0}^{400} \binom{400}{i} (0.75)^{i} (0.25)^{(400-i)} e^{u\, i + d\, (400-i)} \text{ dollars.}$$

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0061 - 1. Price a 30 - day European call option on a stock, using the Black - Scholes Option Pricing Formula.

Assume that the annual drift is 4 %. Assume that the annual volatility is 30 %. Assume that the annual force of interest is 1 %. (That is, \$1, invested risk – free, grows, after one year, to $e^{0.01}$ dollars.)

Assume that the current price is \$2 per share, and that the strike price is also \$2 per share.

We are given that $\sigma_* = 0.3$, $r_* = 0.01$, $K = 2$, and $S_0 = 2$. Then

$$T = \frac{30}{365} \approx 0.0821918,$$

$$\sigma = \sigma_* \sqrt{T} \approx 0.0860073,$$

$$r = r_* T \approx 0.000821918,$$

$$K' = \frac{K}{e^r} \approx \frac{2}{1.0082} \approx 1.99836,$$

$$d_\pm = \frac{\ln(S_0 / K')}{\sigma} \pm \frac{\sigma}{2} \approx 0.0095961 \pm 0.0430037, \text{ so that}$$

$d_+ \approx 0.0525998$ and $d_- \approx -0.0334076$.

Plugging into the Black – Scholes formula, we have
$S_0[\Phi(d_+)] - K'[\Phi(d_-)] \approx (2)(0.2078387) - (1.99836)(0.1941551) = 0.0276856.$

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0061 – 2. Let $C(r_*, \sigma_*, T, S, K)$ denote the Black – Scholes price on a T year European call option, struck at K, with current underlying price S, assuming the annual force of interest is $r_*$, and the annual volatility $\sigma_*$.

a. Compute $\Delta(r_*, \sigma_*, T, S, K) = \frac{\partial}{\partial S}[C(r_*, \sigma_*, T, S, K)]$.

b. Compute $\Gamma(r_*, \sigma_*, T, S, K) = \frac{\partial}{\partial S}[\Delta(r_*, \sigma_*, T, S, K)]$.

c.Compute $V(r_*, \sigma_*, T, S, K) = \frac{\partial}{\partial \sigma_*}[C(r_*, \sigma_*, T, S, K)]$.

a. $\frac{\partial}{\partial S} C = \frac{\partial}{\partial S} S[\Phi(d_+)] - \frac{\partial}{\partial S} K'[\Phi(d_-)] =$

$$\Phi(d_+) + S \frac{\partial \Phi}{\partial d_+} \frac{\partial d_+}{\partial S} - K' \frac{\partial \Phi}{\partial d_-} \frac{\partial d_-}{\partial S} = \Phi(d_+) + S \left(\frac{e^{-d_+{}^2/2}}{\sqrt{2\pi}}\right)\left(\frac{1}{\sigma_* \sqrt{T}\, S}\right) - K e^{-r_* \sqrt{T}} \left(\frac{e^{-d_-{}^2/2}}{\sqrt{2\pi}}\right)\left(\frac{1}{\sigma_* \sqrt{T}\, S}\right) =$$

$$\Phi(d_+) + S \left(\frac{e^{-d_+{}^2/2}}{\sqrt{2\pi}}\right)\left(\frac{1}{\sigma_* \sqrt{T}\, S}\right) - K e^{-r_* \sqrt{T}} \left(\frac{e^{-\left(d_+ - \sigma_* \sqrt{T}\right)^2/2}}{\sqrt{2\pi}}\right)\left(\frac{1}{\sigma_* \sqrt{T}\, S}\right) = \Phi(d_+).$$

b. $\frac{\partial}{\partial S} \Delta = \frac{\partial}{\partial S} \Phi(d_+) = \frac{\partial \Phi}{\partial d_+} \frac{\partial d_+}{\partial S} = \left(\frac{e^{-d_+{}^2/2}}{\sqrt{2\pi}}\right)\left(\frac{1}{\sigma_* \sqrt{T}\, S}\right).$

c. $\frac{\partial}{\partial \sigma_*} C = S \frac{\partial \Phi}{\partial d_+} \frac{\partial d_+}{\partial \sigma_*} - K e^{-r_* \sqrt{T}} \frac{\partial \Phi}{\partial d_-} \frac{\partial d_-}{\partial \sigma_*} =$

$$S \left(\frac{e^{-d_+{}^2/2}}{\sqrt{2\pi}}\right)\left(-\frac{\ln(S_0 / K')}{\sigma_*{}^2 \sqrt{T}} + \frac{\sqrt{T}}{2}\right) - K e^{-r_* \sqrt{T}} \left(\frac{e^{-\left(d_+ - \sigma_* \sqrt{T}\right)^2/2}}{\sqrt{2\pi}}\right)\left(-\frac{\ln(S_0 / K')}{\sigma_*{}^2 \sqrt{T}} - \frac{\sqrt{T}}{2}\right) = \frac{e^{-d_+ \wedge 2/2}}{\sqrt{2\pi}} S \sqrt{T}\, .$$

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0062 – 1. Suppose that, on a certain stock,
the annual drift is 0.04 and the annual volatility is 0.25. Suppose that the current share
price of the stock is \$1. Assume that \$1 invested risk – free for one year grows to $e^{0.015}$ dollars.

a. Using the B - S Option Pricing Formula, price a 0.5 - year call option on the stock with a strike price of \$1.

b. Calibrate the uptick and downtick factors, $e^u$ and $e^d$,
of a 50 – 50 CRR model in which each subperiod is 1 / 6 of a year.

c. Using a 3 - subperiod 50 - 50 CRR model, price a 0.5 - year call option on the stock with a strike price of \$1.

a. We are given that $\sigma_* = 0.25$, $r_* = 0.015$, $K = 1$, $S_0 = 1$, $T = 0.5$. Then
$\sigma = \sigma_* \sqrt{T} \approx 0.1767767$,
$r = r_* T \approx 0.0075$,
$K' = \frac{K}{e^r} \approx 0.99252805$,
$d_\pm = \frac{\ln(S_0 / K')}{\sigma} \pm \frac{\sigma}{2} \approx 0.0424264 \pm 0.08838835$, so that
$d_+ \approx 0.13081475$ and $d_- \approx -0.04596195$.
Plugging into the Black – Scholes formula, we have
$S_0[\Phi(d_+)] - K'[\Phi(d_-)] \approx (1)(0.2202317) - (0.99252805)(0.1921586433) = 0.0295089.$

b. We have that

$\mu_* = \dfrac{0.04}{6} \approx 0.006667$ and $\sigma_* = \dfrac{0.25}{\sqrt{6}} \approx 0.10206.$ Then we solve

$(0.5)\,u + (0.5)\,d = \mu_* \approx 0.006667$ and

$(0.5)\,u - (0.5)\,d = \sigma_* \approx 0.10206$ to obtain

$u \approx 0.108727,\ \ d \approx -0.095393.$

c. $C = e^{-0.15\,(0.5)} \sum_{i=0}^{3} \binom{3}{i} (0.5)^i\,(0.5)^{(3-i)}\left(e^{u\,i+d\,(400-i)}\right)_+ \approx e^{-0.15\,(0.5)}\,(0.96885) \approx 0.9616.$