Financial Mathematics 5002 : Homework 12 (0063 - 0065)

Due on 4 May 2011

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Solutions

0063 - 1. Let S be the set of standard PCRVs.

a. Compute E[7 + 4 S].

b. Compute Var[7 + 4 S].

c. Compute SD[7 + 4 S].

a. E[7 + 4 S] = 7 + 4 E[S] = 7.

b. Var[7 + 4 S] = Var[4 S] = 4^2 Var[S] = 16.

c. SD[7 + 4 S] = 4 SD[S] = 4.

0063 - 2.

a. Compute $E\left[B_{0.25,5}^{0.75,7}\right]$.

b. Compute $Var\left[B_{0.25,5}^{0.75,7}\right]$.

c. Compute $SD\left[B_{0.25,5}^{0.75,7}\right]$.

a. $E\left[B_{0.25,5}^{0.75,7}\right] = 0.75 \times 7 + 0.25 \times 5 = 6.5$.

b. $Var\left[B_{0.25,5}^{0.75,7}\right] = (0.75)\,(0.25)\,(7-5)^2 = 0.75$.

c. $SD\left[B_{0.25,5}^{0.75,7}\right] = \sqrt{(0.75)\,(0.25)\,(7-5)^2} = \sqrt{0.75} \approx 0.866025$.

0063 - 3.

a. Compute $E\left[\sum^{100} B_{0.25,5}^{0.75,7}\right]$.

b. Compute $Var\left[\sum^{100} B_{0.25,5}^{0.75,7}\right]$.

c. Compute $SD\left[\sum^{100} B_{0.25,5}^{0.75,7}\right]$.

a. $E\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 100\, E\left[B_{0.25,5}^{0.75,7}\right] = 650.$

b. $Var\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 100\, Var\left[B_{0.25,5}^{0.75,7}\right] = 75.$

c. $SD\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 10\, SD\left[B_{0.25,5}^{0.75,7}\right] = \sqrt{75} \approx 8.66025.$

0063 - 4.

a. Compute $E\left[\sum^{100} B_{0.25,5+3}^{0.75,7+3}\right]$.

b. Compute $Var\left[\sum^{100} B_{0.25,5+3}^{0.75,7+3}\right]$.

c. Compute $SD\left[\sum^{100} B_{0.25,5+3}^{0.75,7+3}\right]$.

a. $E\left[\sum^{100} B_{0.25,5+3}^{0.75,7+3}\right] = E\left[\sum^{100} \left(3 + B_{0.25,5}^{0.75,7}\right)\right] = 100\, E\left[3 + B_{0.25,5}^{0.75,7}\right] = 950.$

b. $Var\left[\sum^{100} B_{0.25,5+3}^{0.75,7+3}\right] = Var\left[\sum^{100} \left(3 + B_{0.25,5}^{0.75,7}\right)\right] = 100\, Var\left[3 + B_{0.25,5}^{0.75,7}\right] = 75.$

c. $SD\left[\sum^{100} B_{0.25,5+3}^{0.75,7+3}\right] = SD\left[\sum^{100} \left(3 + B_{0.25,5}^{0.75,7}\right)\right] = 10\, SD\left[3 + B_{0.25,5}^{0.75,7}\right] = \sqrt{75} \approx 8.66025.$

0063 - 5.

a. Compute $E\left[300 + \sum^{100} B_{0.25,5}^{0.75,7}\right]$.

b. Compute $Var\left[300 + \sum^{100} B_{0.25,5}^{0.75,7}\right]$.

c. Compute $SD\left[300 + \sum^{100} B_{0.25,5}^{0.75,7}\right]$.

a. $E\left[300+\sum^{100} B_{0.25,5}^{0.75,7}\right] = 100\,E\left[3+B_{0.25,5}^{0.75,7}\right] = 950.$

b. $\mathrm{Var}\left[300+\sum^{100} B_{0.25,5}^{0.75,7}\right] = 100\,\mathrm{Var}\left[3+B_{0.25,5}^{0.75,7}\right] = 75.$

c. $SD\left[300+\sum^{100} B_{0.25,5}^{0.75,7}\right] = 10\,SD\left[3+B_{0.25,5}^{0.75,7}\right] = \sqrt{75} \approx 8.66025.$

0063 - 6.

a. Compute $E\left[\sum^{100} B_{0.25,(3)(5)}^{0.75,(3)(7)}\right]$.

b. Compute $\mathrm{Var}\left[\sum^{100} B_{0.25,(3)(5)}^{0.75,(3)(7)}\right]$.

c. Compute $SD\left[\sum^{100} B_{0.25,(3)(5)}^{0.75,(3)(7)}\right]$.

a. $E\left[\sum^{100} B_{0.25,(3)(5)}^{0.75,(3)(7)}\right] = 3\,E\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 300\,E\left[B_{0.25,5}^{0.75,7}\right] = 1950.$

b. $\mathrm{Var}\left[\sum^{100} B_{0.25,(3)(5)}^{0.75,(3)(7)}\right] = 3^2\,\mathrm{Var}\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 900\,\mathrm{Var}\left[B_{0.25,5}^{0.75,7}\right] = 675.$

c. $SD\left[\sum^{100} B_{0.25,(3)(5)}^{0.75,(3)(7)}\right] = 3\,SD\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 30\,SD\left[B_{0.25,5}^{0.75,7}\right] = \sqrt{675} \approx 25.9808.$

0063 - 7.

a. Compute $E\left[3\sum^{100} B_{0.25,5}^{0.75,7}\right]$.

b. Compute $\mathrm{Var}\left[3\sum^{100} B_{0.25,5}^{0.75,7}\right]$.

c. Compute $SD\left[3\sum^{100} B_{0.25,5}^{0.75,7}\right]$.

a. $E\left[3\sum^{100} B_{0.25,5}^{0.75,7}\right] = 3\,E\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 300\,E\left[B_{0.25,5}^{0.75,7}\right] = 1950.$

b. $\mathrm{Var}\left[3\sum^{100} B_{0.25,5}^{0.75,7}\right] = 3^2\,\mathrm{Var}\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 900\,\mathrm{Var}\left[B_{0.25,5}^{0.75,7}\right] = 675.$

c. $\mathrm{SD}\left[3\sum^{100} B_{0.25,5}^{0.75,7}\right] = 3\,\mathrm{SD}\left[\sum^{100} B_{0.25,5}^{0.75,7}\right] = 30\,\mathrm{SD}\left[B_{0.25,5}^{0.75,7}\right] = \sqrt{675} \approx 25.9808.$

0063 - 8.

a. Compute $\mathrm{E}\left[\exp\left(B_{0.25,5}^{0.75,7}\right)\right]$.

b. Compute $\mathrm{Var}\left[\exp\left(B_{0.25,5}^{0.75,7}\right)\right]$.

c. Compute $\mathrm{SD}\left[\exp\left(B_{0.25,5}^{0.75,7}\right)\right]$.

a. $\mathrm{E}\left[\exp\left(B_{0.25,5}^{0.75,7}\right)\right] = \mathrm{E}\left[B_{0.25,e^5}^{0.75,e^7}\right] = (0.75)\,e^7 + (0.25)\,e^5 \approx 859.578.$

b. $\mathrm{Var}\left[\exp\left(B_{0.25,5}^{0.75,7}\right)\right] = \mathrm{Var}\left[B_{0.25,e^5}^{0.75,e^7}\right] = (0.75)\,(0.25)\left(e^7 - e^5\right)^2 \approx 168\,585.2188.$

c. $\mathrm{SD}\left[\exp\left(B_{0.25,5}^{0.75,7}\right)\right] = \mathrm{SD}\left[B_{0.25,e^5}^{0.75,e^7}\right] = \sqrt{\mathrm{Var}\left[B_{0.25,e^5}^{0.75,e^7}\right]} = \sqrt{(0.75)\,(0.25)}\left(e^7 - e^5\right) \approx 410.5913.$

0063 - 9.

a. Compute $\mathrm{E}\left[B_{0.25,e^5}^{0.75,e^7}\right]$.

b. Compute $\mathrm{Var}\left[B_{0.25,e^5}^{0.75,e^7}\right]$.

c. Compute $\mathrm{SD}\left[B_{0.25,e^5}^{0.75,e^7}\right]$.

a. $\mathrm{E}\left[B_{0.25,e^5}^{0.75,e^7}\right] = (0.75)\,e^7 + (0.25)\,e^5 \approx 859.578.$

b. $\mathrm{Var}\left[B_{0.25,e^5}^{0.75,e^7}\right] = (0.75)\,(0.25)\left(e^7 - e^5\right)^2 \approx 168\,585.2188.$

c. $\mathrm{SD}\left[B_{0.25,e^5}^{0.75,e^7}\right] = \sqrt{\mathrm{Var}\left[B_{0.25,e^5}^{0.75,e^7}\right]} = \sqrt{(0.75)\,(0.25)}\left(e^7 - e^5\right) \approx 410.5913.$

0063 – 10. Compute $\mathrm{E}\left[\prod^{100} \exp\left(B_{0.25,5}^{0.75,7}\right)\right]$.

$$E\left[\prod^{100} \exp\left(B^{0.75,\,7}_{0.25,5}\right)\right] = E\left[\prod^{100} B^{0.75,\,e^7}_{0.25,e^5}\right] = E\left[B^{0.75,\,e^7}_{0.25,e^5}\right]^{100} = \left((0.75)\, e^7 + (0.25)\, e^5\right)^{100} \approx (859.578)^{100}.$$

0063 – 11. Compute $E\left[\exp\left(\sum^{100} B^{0.75,\,7}_{0.25,5}\right)\right]$.

$$E\left[\exp\left(\sum^{100} B^{0.75,\,7}_{0.25,5}\right)\right] = E\left[\prod^{100} \exp\left(B^{0.75,\,7}_{0.25,5}\right)\right] = \left((0.75)\, e^7 + (0.25)\, e^5\right)^{100} \approx (859.578)^{100}, \text{ from 0063 – 10.}$$

0063 – 12. Say $A \in B^{0.2,\,4}_{0.8,1}$ and $B \in B^{0.6,\,8}_{0.4,5}$.

Then the possible values of A + B are 1 + 5, 1 + 8, 4 + 5, and 4 + 8. This is redundant, because 1 + 8 = 4 + 5. Eliminating this redudancy, we have $A + B \in \{6, 9, 12\}$.

Assume A and B are independent. Compute the distribution of A + B, that is, compute each of these :

Pr[A + B = 6], Pr[A + B = 9], and Pr[A + B = 12].

Pr[A + B = 6] = Pr[A = 1 && B = 5] = (0.8) (0.4) = 0.32.

Pr[A + B = 9] = Pr[(A = 1 && B = 8) || (A = 4 && B = 5)] = (0.8) (0.6) + (0.2) (0.4) = 0.56.

Pr[A + B = 12] = Pr[A = 4 && B = 8] = (0.2) (0.6) = 0.12.

0064 – 1. For all integers $i \geq 1$, let $W^{(i)}_1$, $W^{(i)}_2$, ...,
$W^{(i)}_i$ be binary PCRVs, with uptick / downtick probabilities : $p^{(i)}$, $q^{(i)}$.

For all integers $n \geq 1$, note that $p^{(n)} + q^{(n)} = 1$ and let $X^{(n)} = W^{(n)}_1 + \ldots + W^{(n)}_n$.

Say $p^{(n)} \to 0.4$, $E\left[X^{(n)}\right] \to 0.03$, $SD\left[X^{(n)}\right] \to 0.35$, as $n \to \infty$.

Compute $\lim_{n\to\infty} E\left[\left(e^{X^{(n)}} - e\right)_+\right]$.

From the triangular central limit theorem, we have that $X^{(n)} \to \sigma Z + \mu$ in distribution,
so that using the givens, $X^{(n)} \to 0.35 Z + 0.03$. So $\lim_{n\to\infty} E\left[\left(e^{X^{(n)}} - e\right)_+\right] =$

$$E\left[\lim_{n\to\infty}\left(e^{X^{(n)}} - e\right)_+\right] = E\left[\left(e^{0.35 Z+0.03} - e\right)_+\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{0.35 x+0.03} - e\right)_+ e^{-x^2/2}\,dx =$$

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{97}{35}}^{\infty}\left(e^{0.35 x+0.03} - e\right) e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{\frac{97}{35}}^{\infty} e^{-x^2/2+0.35 x+0.03}\,dx - \frac{e}{\sqrt{2\pi}}\int_{\frac{97}{35}}^{\infty} e^{-x^2/2}\,dx =$$

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{97}{35}}^{\infty} e^{-1/2 (x-0.35)^2+\frac{73}{800}}\,dx - e\,\Phi\left(-\frac{97}{35}\right) = \frac{e^{0.09125}}{\sqrt{2\pi}}\int_{\frac{97}{35}-0.35}^{\infty} e^{-x^2/2}\,dx - e\,\Phi\left(-\frac{97}{35}\right) =$$

$$e^{0.09125}\,\Phi\left(0.35 - \frac{97}{35}\right) - e\,\Phi\left(-\frac{97}{35}\right) \approx e^{0.09125}\,\Phi(-2.42143) - e\,\Phi(-2.77143).$$

0065 - 1.

a. Using the Black - Scholes Option Pricing Formula, price a 0.25 - year call option on a stock with current share price \$2, with strike price \$2 .10, with annual volatility 0.31, and with annual risk - free rate ln (1.04).

b. Let $S_0 = 2$, $K = 2.1$, $\sigma_* = 0.31$, and $r_* = \ln(1.04)$. Let $\sigma = \sigma_* \Big/ \sqrt{4}$,

$r = r_* \Big/ \sqrt{4}$, and $\nu = r - \left[\sigma^2 / 2\right]$. Compute $\dfrac{e^{-r}}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\left(S_0\, e^{\sigma x+\nu} - K\right)_+ e^{-x^2/2}\,dx$.

a. We are given that $\sigma_* = 0.31$, $r_* = \ln(1.04)$, $K = 2.1$, and $S_0 = 2$. Then
$T = 0.25$,
$\sigma = \sigma_* \sqrt{T} \approx 0.155$,
$r = r_*\, T \approx 0.00980518$,
$K' = \dfrac{K}{e^r} \approx \dfrac{2.1}{1.04} \approx 2.07951$,
$d_\pm = \dfrac{\ln(S_0 / K')}{\sigma} \pm \dfrac{\sigma}{2} \approx -0.2515 \pm 0.0775$, so that
$d_+ \approx -0.174$ and $d_- \approx -0.329$.
Plugging into the Black – Scholes formula, we have
$S_0[\Phi(d_+)] - K'[\Phi(d_-)] = 2[\Phi(-0.174)] - 2.07951[\Phi(-0.329)]$.

b. $\dfrac{e^{-r}}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\left(S_0\, e^{\sigma x+\nu} - K\right)_+ e^{-x^2/2}\,dx =$

$$\frac{e^{-r}}{\sqrt{2\pi}}\int_{\frac{\ln(K/S_0)-\nu}{\sigma}}^{\infty}\left(S_0\, e^{\sigma x+\nu} - K\right) e^{-x^2/2}\,dx = S_0\,\Phi(d_+) - K\, e^{-r}\,\Phi(d_-) = 2[\Phi(-0.174)] - 2.07951[\Phi(-0.329)].$$
