Financial Mathematics 5002 : Homework 3 (0041)

Due on 9 Februrary 2011

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Solutions

0041 – 1. Fix $r > 0$.

Let $D = (0, r) \times (0, 2\pi)$ and $E = \left(\left\{(x, y) \middle| x^2 + \frac{y^2}{9} < r^2\right\}\right) \setminus ([0, r) \times \{0\})$.

Define $f : D \to E$ by $f(s, t) = (s\,\mathrm{Cos}[t], 3\,s\,\mathrm{Sin}[t])$. Then f is a smooth bijection.

$$f'(x, t) = \begin{pmatrix} \mathrm{Cos}[t] & -s\,\mathrm{Sin}[t] \\ 3\,\mathrm{Sin}[t] & 3\,s\,\mathrm{Cos}[t] \end{pmatrix}.$$

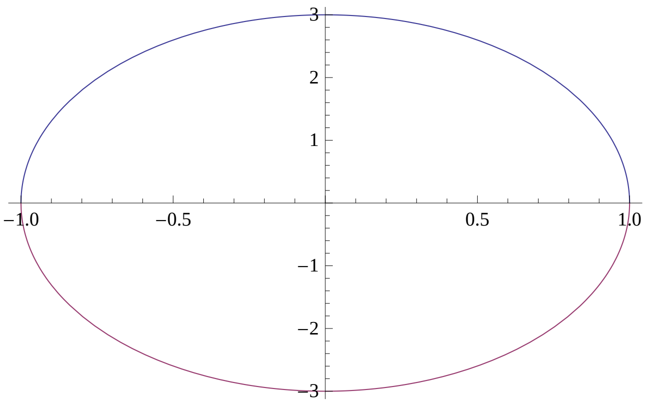
$$\mathrm{Area}(E) = \int\int_D \left|\det f'(s, t)\right| ds\, dt = \int_0^{2\pi}\int_0^r \left|\det f'(s, t)\right| ds\, dt.$$

a. Finish this computation.

b. Graph E.

a. $$\int_0^{2\pi}\int_0^r \left|\det f'(s, t)\right| ds\, dt = \int_0^{2\pi}\int_0^r \left|3\,s\,\mathrm{Cos}[t]^2 + 3\,s\,\mathrm{Sin}[t]^2\right| ds\, dt = \int_0^{2\pi}\int_0^r 3\,s\, ds\, dt = \int_0^{2\pi} \frac{3\,r^2}{2}\, dt = 3\,\pi\,r^2.$$

b. Observe that E is just an ellipse centered at 0. The picture below is for $r = 1$.



0041 - 2. Let $D = (0, 5) \times (0, \pi) \times (0, \pi/2)$.

Define $\psi : D \to \mathbb{R}^3$ by $\psi(r, \theta, \phi) = (r\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\theta], r\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta], 2\,r\,\mathrm{Sin}[\phi])$.

Let $E := \psi(D)$.

Then $\psi : D \to E$ is bijective and smooth and has continuous extension to the closure of D.

Compute the volume of E by computing $\int_E 1\, dx\, dy\, dz$ via the change of variables formula.

We compute that $J(r, \theta, \phi) = \det\begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\phi] & -r\,\text{Cos}[\phi]\,\text{Sin}[\theta] & -r\,\text{Cos}[\theta]\,\text{Sin}[\phi] \\ \text{Cos}[\phi]\,\text{Sin}[\theta] & r\,\text{Cos}[\theta]\,\text{Cos}[\phi] & -r\,\text{Sin}[\theta]\,\text{Sin}[\phi] \\ 2\,\text{Sin}[\phi] & 0 & 2\,r\,\text{Cos}[\phi] \end{pmatrix} =$

$2\,r^2\,\text{Cos}[\theta]^2\,\text{Cos}[\phi]^3 + 2\,r^2\,\text{Cos}[\phi]^3\,\text{Sin}[\theta]^2 + 2\,r^2\,\text{Cos}[\theta]^2\,\text{Cos}[\phi]\,\text{Sin}[\phi]^2 + 2\,r^2\,\text{Cos}[\phi]\,\text{Sin}[\theta]^2\,\text{Sin}[\phi]^2 =$ $2\,r^2\,\text{Cos}[\phi]$. Therefore, by the change of variables formula, we have that

$$\int_E 1\, dx\, dy\, dz =$$

$$\int_0^5 \int_0^\pi \int_0^{\pi/2} \left| J(r, \theta, \phi) \right| d\phi\, d\theta\, dr = \int_0^5 \int_0^\pi \int_0^{\pi/2} 2\,r^2\,\text{Cos}[\phi]\, d\phi\, d\theta\, dr = \int_0^5 \int_0^\pi 2\,r^2\, d\theta\, dr = \int_0^5 2\,\pi\,r^2\, dr = \frac{250\,\pi}{3}.$$