Financial Mathematics 5002 : Homework 5 (0044)

Due on 23 Februrary 2011

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Solutions

0044 – 1. Compute $\int \left(x^3 - x^2\right) e^{-\frac{x^2}{2}} \, dx$.

Using the results from class, $\int \left(x^3 - x^2\right) e^{-\frac{x^2}{2}} \, dx = \int x^3 e^{-\frac{x^2}{2}} \, dx - \int x^2 e^{-\frac{x^2}{2}} \, dx = e^{-\frac{x^2}{2}} \left(-2 + x - x^2\right) - \sqrt{2\pi} \; \Phi(x)$.

0044 – 2. Compute $\int_2^\infty e^{-\frac{x^2}{2}} \, dx$.

$\int_2^\infty e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \; \Phi(-2)$.

0044 – 3. Compute $\int_9^\infty e^x e^{-\frac{x^2}{2}} \, dx$.

We complete the square : $\int_9^\infty e^x e^{-\frac{x^2}{2}} \, dx = \int_9^\infty e^{-\frac{(x-1)^2}{2} + \frac{1}{2}} \, dx = e^{\frac{1}{2}} \int_9^\infty e^{-\frac{(x-1)^2}{2}} \, dx =$

$e^{\frac{1}{2}} \int_8^\infty e^{-\frac{u^2}{2}} \, du = \sqrt{2\pi e} \; \Phi(-8)$. Note that this shows that $\int_y^\infty e^x e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi e} \; \Phi(-y+1)$.

0044 – 4. Compute $\int_4^\infty \left(3 e^x - 3 e^4\right) e^{-\frac{x^2}{2}} \, dx$.

From the previous problems,

$\int_4^\infty \left(3 e^x - 3 e^4\right) e^{-\frac{x^2}{2}} \, dx = 3 \int_4^\infty e^x e^{-\frac{x^2}{2}} \, dx - 3 e^4 \int_4^\infty e^{-\frac{x^2}{2}} \, dx = 3 \sqrt{2\pi e} \; \Phi(-3) - 3 e^4 \sqrt{2\pi} \; \Phi(-4)$.

0044 – 5. Compute $\int_{-\infty}^\infty \left(3 e^x - 3 e^4\right)_+ e^{-\frac{x^2}{2}} \, dx$.

$\left(3 e^x - 3 e^4\right) \geq 0$ when $x \geq 4$,

so that $\int_{-\infty}^\infty \left(3 e^x - 3 e^4\right)_+ e^{-\frac{x^2}{2}} \, dx = \int_4^\infty \left(3 e^x - 3 e^4\right) e^{-\frac{x^2}{2}} \, dx = 3 \sqrt{2\pi e} \; \Phi(-3) - 3 e^4 \sqrt{2\pi} \; \Phi(-4)$, from 0044 – 4.

0044 – 6. Compute $\int_{-\infty}^{\infty} (2\,e^x - 9)_+ \, e^{-\frac{x^2}{2}} \, dx$.

$(2\,e^x - 9) \geq 0$ when $x \geq \ln(9/2)$,

so that $\int_{-\infty}^{\infty} (2\,e^x - 9)_+ \, e^{-\frac{x^2}{2}} \, dx = \int_{\ln(9/2)}^{\infty} (2\,e^x - 9)_+ \, e^{-\frac{x^2}{2}} \, dx = 2\sqrt{2\pi e}\; \Phi(-\ln(9/2) + 1) - 9\sqrt{2\pi}\; \Phi(-\ln(9/2))$

0044 – 7. Compute $\int_{-\infty}^{\infty} x^4 \, e^{-\frac{x^2}{2}} \, dx$ and $\int_{-\infty}^{\infty} x^5 \, e^{-\frac{x^2}{2}} \, dx$.

$\int_{-\infty}^{\infty} x^4 \, e^{-\frac{x^2}{2}} \, dx = 3\sqrt{2\pi}$ and $\int_{-\infty}^{\infty} x^5 \, e^{-\frac{x^2}{2}} \, dx = 0$, by results from class (even and odd powers, respectively).

0044 - 8.

a. Compute $\int_{-\infty}^{\infty} x^3 \, e^{-\frac{x^2}{2}} \, dx$.

b. Compute $\int_{-5}^{7} x^3 \, e^{-\frac{x^2}{2}} \, dx$.

c. Compute $\int_{-\infty}^{\infty} (x^6 - x) \, e^{-\frac{x^2}{2}} \, dx$.

d. Compute $\int_{-\infty}^{\infty} (2\,x^2 + 4\,x + 6) \, e^{-\frac{x^2}{2}} \, dx$.

a. $\int_{-\infty}^{\infty} x^3 \, e^{-\frac{x^2}{2}} \, dx = 0$.

b. $\int_{-5}^{7} x^3 \, e^{-\frac{x^2}{2}} \, dx = \left[e^{-\frac{x^2}{2}} \, (-2 - x^2) \right]_{-5}^{7} = -51\, e^{-49/2} + 27\, e^{-25/2}$

c. $\int_{-\infty}^{\infty} (x^6 - x) \, e^{-\frac{x^2}{2}} \, dx = \int_{-\infty}^{\infty} x^6 \, e^{-\frac{x^2}{2}} \, dx - \int_{-\infty}^{\infty} x \, e^{-\frac{x^2}{2}} \, dx = 15\sqrt{2\pi}$.

d. $\int_{-\infty}^{\infty} (2\,x^2 + 4\,x + 6) \, e^{-\frac{x^2}{2}} \, dx = 2\int_{-\infty}^{\infty} x^2 \, e^{-\frac{x^2}{2}} \, dx + 4\int_{-\infty}^{\infty} x \, e^{-\frac{x^2}{2}} \, dx + 6\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = 2\sqrt{2\pi} + 6\sqrt{2\pi} = 8\sqrt{2\pi}$.

0044 - 9.

a. Compute $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-5x+3} \, e^{-\frac{x^2}{2}} \, dx$.

b. Compute $\frac{1}{\sqrt{2\pi}}\int_{-2}^{7} e^{-5x+3}\, e^{-\frac{x^2}{2}}\, dx$.

c. Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(e^{-5x+3}-1\right) e^{-\frac{x^2}{2}}\, dx$.

d. Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(e^{-5x+3}-1\right)_+ e^{-\frac{x^2}{2}}\, dx$.

a. We complete the square and use the substution u = x − 5 : $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-5x+3}\, e^{-\frac{x^2}{2}}\, dx = \frac{e^{31/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\, du = e^{31/2}$.

b. $\frac{1}{\sqrt{2\pi}}\int_{-2}^{7} e^{-5x+3}\, e^{-\frac{x^2}{2}}\, dx = \frac{e^{31/2}}{\sqrt{2\pi}}\int_{3}^{12} e^{-\frac{u^2}{2}}\, du = e^{31/2}\,(\Phi(12)-\Phi(3))$.

c. $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(e^{-5x+3}-1\right) e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-5x+3}\, e^{-\frac{x^2}{2}}\, dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = e^{31/2}-1$.

d. $e^{-5x+3}-1 \geq 0$ when $x \leq \frac{3}{5}$. Therefore,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(e^{-5x+3}-1\right)_+ e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{3}{5}} \left(e^{-5x+3}-1\right) e^{-\frac{x^2}{2}}\, dx = \frac{e^{31/2}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{28}{5}} e^{-\frac{u^2}{2}}\, du - \Phi\left(\frac{3}{5}\right) = e^{31/2}\,\Phi\left(\frac{28}{5}\right) - \Phi\left(\frac{3}{5}\right).$$