Financial Mathematics 5002 : Homework 6 (0045-0046)

Due on 9 March 2011

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Solutions

0045 – 1. Let X and Y be identically distributed PCRVs. Assume Pr[X = 1] = 0.2 and Pr[X = 9] = 0.8. Let $\rho \in [-1, 1]$. Assume $E[XY] = (E[Y])^2 + \rho\,(Var[Y])$.

a. Compute E[X + Y].

b. Compute Var[X + Y]. Your answer will depend on ρ.

c. Find the value of $\rho \in [-1, 1]$ where Var[X + Y] is minimized.

d. Find the value of $\rho \in [-1, 1]$ where Var[X + Y] is maximized.

a. By linearity of expectation, and since X and Y are identicaly distributed, we have E[X + Y] = E[X] + E[Y] = 2 E[X] = 2 (1 * 0.2 + 9 * 0.8) = 14.8.

b. We compute that $Var[X] = Var[Y] = 0.2\,(1-7.4)^2 + 0.8\,(9-7.4)^2 = 10.24$.

Furthermore, $Cov[X, Y] = E[X\,Y] - E[X]\,E[Y] = (E[Y])^2 + \rho\,(Var[Y]) - E[X]\,E[Y] = \rho\,(Var[Y]) = 10.24\,\rho$.

Then Var[X + Y] = Var[X] + Var[Y] + 2 Cov[X, Y] = 2 (Var[X] + Cov[X, Y]) = 2 (10.24 + 10.24 ρ) = 20.48 (1 + ρ).

c. Since Var[X + Y] is a linear function of ρ, it is easy to see that Var[X + Y] is minimized at $\rho = -1$.

d. Since Var[X + Y] is a linear function of ρ, Var[X + Y] is maximized at $\rho = 1$.

0045 - 2. Let X be a PCRV such that Pr[X = 1] = 0.2, Pr[X = 3] = 0.2, and Pr[X = 8] = 0.6. Find E[X], Var[X], and SD[X].

E[X] = 1*0.2 + 3*0.2 + 8*0.6 = 5.6.

$Var[X] = 0.2\,(1-5.6)^2 + 0.2\,(3-5.6)^2 + 0.6\,(8-5.6)^2 = 9.04$.

$SD[X] = \sqrt{Var[X]} = \sqrt{9.04} \approx 3.0067$.

0045 - 3. Let X be a binary PCRV such that Pr[X = 4] = 0.85 and Pr[X = 18] = 0.15. Find the mean, the variance, and the standard deviation of X.

E[X] = 4*0.85 + 18*0.15 = 6.1.

$\text{Var}[X] = 0.85\,(4 - 6.1)^2 + 0.15\,(18 - 6.1)^2 = 24.99$

$\text{SD}[X] = \sqrt{\text{Var}[X]} = \sqrt{24.99} \approx 5.$

0045 - 4.

$\text{Let } W = \begin{cases} -2 \text{ if } 0 \le \omega < 0.7, \\ -3 \text{ if } 0.7 \le \omega \le 1. \end{cases}$

$\text{Let } X = \begin{cases} 4 \text{ if } 0 \le \omega < 0.7, \\ 6 \text{ if } 0.7 \le \omega \le 1. \end{cases}$

$\text{Let } Y = \begin{cases} 10^7 \text{ if } 0 \le \omega < 0.7, \\ 10^8 \text{ if } 0.7 \le \omega \le 1. \end{cases}$

$\text{Let } Z = \begin{cases} 3 \text{ if } 0 \le \omega < 0.7001, \\ 9 \text{ if } 0.7001 \le \omega \le 1. \end{cases}$

Compute :

a. Corr[W, X].

b. Corr[W, Y].

c. Corr[X, Y].

d. Corr[X, Z].

e. Corr[Y, Z].

Since $\text{Corr}[A, B] = \dfrac{\text{Cov}[A, B]}{\text{SD}[A]\,\text{SD}[B]}$ and $\text{Cov}[A, B] = E[AB] - E[A]\,E[B]$, we need to compute Var[W], Var[X], Var[Y], and Var[Z], as well as E and Cov of every pair except W and Z.

Using the formulas and that X = -2 W, we compute that

E[W] = -2.3

E[X] = 4.6

$E[Y] = 3.7 * 10^7$

E[Z] = 4.7994

Var[W] = 0.21

Var[X] = 0.84

$\text{Var}[Y] = 1.701 * 10^{15}$

$Var[Z] = 7.5586$

$E[W X] = -11$

$E[W Y] = -1.04 * 10^8$

$E[X Y] = 2.08 * 10^8$

$E[X Z] = 24.5964$

$E[Y Z] = 2.9094 * 10^8$

$Cov[W, X] = -0.42$

$Cov[W, Y] = -1.89 * 10^7$

$Cov[X, Y] = 3.78 * 10^7$

$Cov[X, Z] = 2.5192$

$Cov[Y, Z] = 1.133622 * 10^8$

so that

$Corr[W, X] = -1$

$Corr[W, Y] = -1$

$Corr[X, Y] = 1$

$Corr[X, Z] = 0.9998$

$Corr[Y, Z] = 0.9998$

0045 – 5. Let C_1 and C_2 be our standard coin flipping random variables. Let $\rho \in [-1, 1]$. Find constants a, b, c, $d \in \mathbb{R}$ such that if we define $X = a C_1 + b C_2$ and $Y = c C_1 + d C_2$ then we have $SD[X] = 1$, $SD[Y] = 5$, and $Corr[X, Y] = \rho$, and such that at least one of a, b, c, d is zero.

$Corr[X, Y] = \rho = \frac{Cov[X, Y]}{SD[X]\, SD[Y]} = \frac{Cov[X, Y]}{5}$, so that $Cov[X, Y] = 5 \rho$. On the other hand, since $E[C_1] = E[C_2] = E[C_1 C_2] = 0$ and by linearity of expectation,

$$Cov[X, Y] = E[X Y] - E[X] E[Y] = E[(a C_1 + b C_2)(c C_1 + d C_2)] - E[a C_1 + b C_2] E[c C_1 + d C_2] =$$
$$E\left[a c C_1^2 + b d C_2^2 + (a d + b c) C_1 C_2\right] = a c E\left[C_1^2\right] + b d E\left[C_2^2\right] + (a d + b c) E[C_1 C_2] = a c + b d = 5 \rho.$$

Lastly, $SD[X] =$
1 tells us that $Var[X] = E\left[X^2\right] - E[X]^2 = E\left[(a C_1 + b C_2)^2\right] = E\left[a^2 C_1^2 + b^2 C_2^2 + 2 a b C_1 C_2\right] = a^2 + b^2 = 1$, and $SD[Y] = 5$ that $Var[Y] = E\left[Y^2\right] - E[Y]^2 = E\left[(c C_1 + d C_2)^2\right] = E\left[c^2 C_1^2 + d^2 C_2^2 + 2 c d C_1 C_2\right] = c^2 + d^2 = 25$.

Letting $a = 0$, we have that
$b\, d = 5\rho$,
$b^2 = 1$, and
$c^2 + d^2 = 25$, from which we conclude that a solution is given by

$a = 0$, $b = 1$, $c = \sqrt{25 - 25\rho^2}$ $d = 5\rho$.

0045 - 6. Show an example of two PCRVs X and Y such that Cov[X, Y] = 0, but such that X and Y are not independent.

We want $\text{Cov}[X, Y] = E[X\,Y] - E[X]\,E[Y] =$

$$0. \text{ Let } X(t) = \begin{cases} -1 \text{ if } 0 \le t \le 1/3 \\ 0 \text{ if } 1/3 < t \le 2/3 \\ 1 \text{ if } 2/3 < t \le 1 \end{cases} \text{ and } Y(t) = \begin{cases} 0 \text{ if } 0 \le t \le 1/3 \\ 1 \text{ if } 1/3 < t \le 2/3 \\ 0 \text{ if } 2/3 < t \le 1 \end{cases}. \text{ Then } E[X] = 0 \text{ and } E[X\,Y] = 0,$$

but X and Y are not independent, since $P[X \le 0 \text{ and } Y \le 0] = 1/3$,
but $P[X \le 0] = 2/3$ and $P[Y \le 0] = 2/3$, so $P[X \le 0 \text{ and } Y \le 0] \ne P[X \le 0]\, P[Y \le 0]$.

0045 - 7. Let X be a PCRV.

a. Let v = Var[X]. What is Cov[-3 X, 2 X]? Your answer will be a (very simple) formula involving v.

b. What is Corr[-3 X, 2 X]?

a. $\text{Cov}[-3X, 2X] = E[(-3X)(2X)] - E[-3X]\,E[2X] = (-6)\left(E\left[X^2\right] - E[X]^2\right) = -6\,\text{Var}[X] = -6\,v$.

b. Observe that $\text{Var}[a\,X] = E\left[(a\,X)^2\right] - E[a\,X]^2 = a^2\left(E\left[X^2\right] - E[X]^2\right) =$

$$a^2\,\text{Var}[x]. \text{ Then } \text{Corr}[-3X, 2X] = \frac{\text{Cov}[-3X, 2X]}{\sqrt{\text{Var}[-3X]}\,\sqrt{\text{Var}[2X]}} = \frac{-6\,v}{\sqrt{9\,\text{Var}[X]}\,\sqrt{4\,\text{Var}[X]}} = \frac{-6\,v}{6\,v} = -1.$$

0045 - 8. You manage a portfolio with three assets that, today, have per share prices of $1, $1, and $1. Denote their prices, one year from now, by the PCRVs $B, $P, and $Q, respectively. Your market analyst tells you :

E[B] = 1.01, E[P] = 1.16, E[Q] = 1.08, Var[B] = 0, Var[P] = 0.3, Var[Q] = 0.1, Cov[B, P] = 0, Cov[B, Q] = 0, and Cov[P, Q] = 0.25.

You plan to buy b, p, and q shares respectively, but you only have $5 to invest, and you wish to achieve a 4 % expected return. Find the portfolio that minimizes variance.

Our setup is as follows :

$b + p + q - 5 = 0,$
$1.01\,b + 1.16\,p + 1.08\,q - 5.2 = 0,$

and we want to minimize
$\mathrm{Var}[b + p + q] = \mathrm{Var}[b\,B] + \mathrm{Var}[p\,P] + \mathrm{Var}[q\,Q] + 2\,\mathrm{Cov}[p\,P,\ q\,Q] = 0.3\,p^2 + 0.1\,q^2 + 0.5\,p\,q.$

This is a Lagrange multiplier problem (with two constraints), so

$b + p + q - 5 = 0,$
$1.01\,b + 1.16\,p + 1.08\,q - 5.2 = 0,$
$(0,\ 0.6\,p + 0.5\,q,\ 0.2\,q + 0.5\,p) - \lambda_1\,(1,\ 1,\ 1) - \lambda_2\,(1.01,\ 1.16,\ 1.08) = 0.$

Solving, we obtain

(b, p, q) = (3.1373, 0.2451, 1.6177).

0046 – 1. Let $X = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 7 & 6 \\ 0 & 6 & 7 \end{pmatrix}$.

a. Find a 3 x 3 matrix A such that $AA^t = X$.

b. Does there exist a symmetric 3 x 3 matrix S such that $S^2 = X$?

a. One such solution is $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \sqrt{6} & 0 \\ 0 & \sqrt{6} & 1 \end{pmatrix}$(obtained by assuming A was lower triangular).

b. $\det\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 7-\lambda & 6 \\ 0 & 6 & 7-\lambda \end{pmatrix} = 6 - 26\,\lambda + 15\,\lambda^2 - \lambda^3 = 0$ for $\lambda = 0.27,\ 1.68,$

and 13.04. Since the all the eigenvalues of X are positive, there does exist such an S. Alternatively, see part a.

0046 – 2. Let $X = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 6 & 6 \\ 0 & 6 & 7 \end{pmatrix}$.

Does there exist a symmetric 3 x 3 matrix S such that $S^2 = X$?

$\det\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 6-\lambda & 6 \\ 0 & 6 & 7-\lambda \end{pmatrix} = -1 - 18\,\lambda + 14\,\lambda^2 - \lambda^3 = 0$ for $\lambda = -0.53,\ 1.49,$

and 12.56. Since not all the eigenvalues of X are positive, there does not exist such an S.

0046 – 3. Let $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$, $C = \begin{pmatrix} 3 & -4 & -7 \\ -1 & 2 & 4 \\ 6 & -11 & -21 \end{pmatrix}$.

a. Is X positive definite?

b. Is X positive semidefinite?

c. Does CXC^{-1} have only positive eigenvalues?

d. Does CXC^{-1} have only nonnegative eigenvalues?

e. Does there exist a symmetric matrix S such that $S^2 = CXC^{-1}$?

a. $\det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 8-\lambda \end{pmatrix} = -16\lambda + 10\lambda^2 - \lambda^3 = 0$ for $\lambda = 0,\ 2,$ and 8. Therefore, X is not positive definite because 0 is an eigenvalue.

b. X is positive semidefinite because all the eigenvalues are nonnegative.

c. C is invertible ($\det C = 1$), so the eigenvalues of CXC^{-1} are the same as those for X. Therefore, CXC^{-1} does not have only positive eigenvalues.

d. CXC^{-1} has only nonnegative eigenvalues (see part c)

e. Since not all the eigenvalues of X (and therefore CXC^{-1}) are positive, there does not exist such an S.

0046 – 4. Let $X = \begin{pmatrix} 14 & 5 & -1 \\ 5 & 5 & -2 \\ -1 & -2 & 1 \end{pmatrix}$.

a. Find a 3 x 3 matrix A such that $AA^t = X$ and such that A is lower triangular.

b. Find a 3 x 3 matrix B such that $BB^t = X$ and such that B is lower triangular.

c. Find a 3 x 3 matrix C such that $C^t C = X$ and such that C is upper triangular.

d. Find a 3 x 3 matrix D such that $D^t D = X$ and such that D is upper triangular.

e. Does there exist a symmetric 3 x 3 matrix S such that $S^2 = X$?

a. $A = \begin{pmatrix} -\sqrt{14} & 0 & 0 \\ -\frac{5}{\sqrt{14}} & -3\sqrt{\frac{5}{14}} & 0 \\ \frac{1}{\sqrt{14}} & \frac{23}{3\sqrt{70}} & -\frac{2}{3\sqrt{5}} \end{pmatrix}$.

b. $B = \begin{pmatrix} -2 & 0 & 0 \\ -3 & -1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$.

c. Let $C = A^t$.

d. Let $D = B^t$.

e. $\det\begin{pmatrix} 14-\lambda & 5 & -1 \\ 5 & 5-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} = 4 - 59\lambda + 20\lambda^2 - \lambda^3 = 0$ when $\lambda = 0.069,\ 3.51,$

and 16.42. Therefore, since all the eigenvalues are positive, there does exist such an S.