This list of problems is not guaranteed to be an absolutely complete review. For completeness you must also make sure that you know how to do all of the homework assigned in the course up to and including that due on November 4, and all the worksheet problems through week 9, with emphasis on weeks 5–9. And of course don’t forget basic skills like using reduced row echelon form to solve linear equations!

To study under typical test conditions, here are the ground rules. You may use matrix addition and multiplication functions on your calculator along with \texttt{det} and \texttt{rref} functions but you must indicate clearly where and how you used these calculator functions—otherwise your answers would be considered unjustified and get little credit. Fancier calculator functions, e.g., those finding eigenvalues and eigenvectors, are not allowed to justify your work. All algebra and calculus work (except for the matrix functions specifically allowed above) must be performed by hand and shown clearly.

Below we supply answers and hints about methods but mostly do not supply fully worked solutions.

**Problem 1**

Solve the initial value problem \(y'' + 49y = 0, \ y(0) = 1, \ y'(0) = 2\), expressing the answer in the form \(A \cos(\omega t - \delta)\) where \(A, \omega\) and \(\delta\) are positive numbers and furthermore \(\delta < 2\pi\). Give numerical approximations for \(A, \omega\) and \(\delta\) accurate to four decimal places.

**Answer.** \(y = 1.0400 \cos(7t - 0.2783)\).

**Problem 2**

A mass of 5 kg is suspended from a high ceiling by a spring. When hanging motionless, the mass stretches the spring 2 meters beyond its natural length. Let \(y\) denote the height of the mass above the equilibrium point. Initially the mass is pulled down \(1/2\) meter below the equilibrium point and released with a velocity of 1 m/sec upward. The mass encounters air resistance of \(1.5 \text{ nt/(m/sec)}\). A force of \(7 \cos(2t)\) nt is applied to the weight. Write out the initial value problem determining the motion of the mass. Is this system underdamped, overdamped or critically damped? Use 10 m/sec\(^2\) as the acceleration of gravity. You do not have to solve the initial value problem.

**Answer.** The template is \(m y'' + b y' + ky = F(t), \ y(0) = y_0, \ y'(0) = v_0\). We are given \(m = 5 \text{ kg}, \ b = 1.5 \text{ nt/(m/sec)}, \ F(t) = 7 \cos(2t), \ y_0 = 0.5 \text{ m} \) and \(v_0 = 1 \text{ m/sec}\). The weight of the mass is \((5 \text{ kg}) \cdot (10 \text{ m/sec}^2) = 50 \text{ nt, so } \ k = (50 \text{ nt})/(2 \text{ m}) = 25 \text{ nt/m}\. \) Thus the initial value problem is \(5y'' + 1.5y' + 25y = 7 \cos(2t), \ y(0) = -0.5, \ y'(0) = 1\. \) Because \(b^2 - 4km = 1.5^2 - 4 \cdot 5 \cdot 25 < 0\), the system is UNDERDAMPED.

**Problem 3**

Solve the following IVP’s:

\[
\begin{align*}
y'' + 5y' + 6y &= 0, \ y(0) = -2, \ y'(0) = 11 \\
y'' + 6y' + 9y &= 0, \ y(0) = 4, \ y'(0) = -23 \\
y'' + 4y' + 29y &= 0, \ y(0) = -2, \ y'(0) = 39
\end{align*}
\]
Answer.
\[ y = 5e^{-2t} - 7e^{-3t} \]
\[ y = 4e^{-3t} - 11te^{-3t} \]
\[ y = -2e^{-2t}\cos(5t) + 7e^{-2t}\sin(5t) \]

Answer. \( y = C_1/t^2 + C_2/t^3 \)

**Problem 4**

A crash-dummy weighs 224 lbs (and thus has mass \( 224/32 = 7 \) slugs). The crash-dummy encounters air resistance of 1.4 lbs/(ft/sec). The crash-dummy is tossed off a building 200 ft high, initially with a velocity of 21 ft/sec upward. Let \( y \) denote the height of the crash-dummy above the ground. Set up and solve the initial value problem for the vertical motion of the crash dummy before it hits the ground. Use your calculator to find the time the crash dummy hits the ground, accurate to four decimal places. Use the method of undetermined coefficients to solve the initial value problem. Also do the problem without air resistance and compare the times of impact.

Answer.
setup: \( 7y'' + 1.4y' = -224 \), \( y(0) = 200 \), \( y'(0) = 21 \).
after simplification: \( y'' + 0.2y' = -32 \), \( y(0) = 200 \), \( y'(0) = 21 \).
\[ y = 0 \Rightarrow t = 4.6942 \]
Without air-resistance: \( y = -16t^2 + 200 + 21t \)
\[ y = 0 \Rightarrow t = 4.2522 \]
Ignoring air-resistance the crash-dummy hits the ground sooner.

**Problem 5**

Find particular solutions by the method of undetermined coefficients for each of the following equations:

\[ y'' + 3y' + 2y = te^{3t} \]
\[ y'' + 3y' + 2y = te^{-2t} \]
\[ y'' + 4y = 3\cos(2t) \]

Answer.
\[ y = \left(\frac{t}{20} - \frac{9}{400}\right)e^{3t} \]
\[ y = \left(-\frac{t^2}{2} - t\right)e^{-2t} \]
\[ \frac{3}{4}t\sin(2t) \]
Problem 6

Find the eigenvalues and eigenvectors for

\[
\begin{bmatrix}
11 & 15/2 \\
-10 & -9
\end{bmatrix}
\]

Given that the eigenvalues are \( \lambda = 1, 3, 5 \), find the eigenvectors for

\[
\begin{bmatrix}
2 & 0 & 1 \\
3 & -4 & 4 \\
7 & -15 & 11
\end{bmatrix}.
\]

Then use the eigenvectors and eigenvalues you have found to diagonalize the matrices.

Answer. For the first matrix, eigenvalue-eigenvector pairs are

\[6 \leftrightarrow \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad -4 \leftrightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}\]

and so we get a diagonalization \( AP = PD \) like this:

\[
\begin{bmatrix}
11 & 15/2 \\
-10 & -9
\end{bmatrix}
\begin{bmatrix}
3 & -1 \\
-2 & 2
\end{bmatrix} =
\begin{bmatrix}
3 & -1 \\
-2 & 2
\end{bmatrix}
\begin{bmatrix}
6 & 0 \\
0 & -4
\end{bmatrix}.
\]

For the second matrix eigenvalue-eigenvector pairs are

\[
\begin{bmatrix}
-5 \\
1 \\
5
\end{bmatrix} \leftrightarrow 1, \quad \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \leftrightarrow 3, \quad \begin{bmatrix}
3 \\
5 \\
9
\end{bmatrix} \leftrightarrow 5
\]

and so we get a diagonalization \( AP = PD \) like this:

\[
\begin{bmatrix}
2 & 0 & 1 \\
3 & -4 & 4 \\
7 & -15 & 11
\end{bmatrix}
\begin{bmatrix}
-5 & 1 & 3 \\
1 & 1 & 5 \\
5 & 1 & 9
\end{bmatrix} =
\begin{bmatrix}
-5 & 1 & 3 \\
1 & 1 & 5 \\
5 & 1 & 9
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 5
\end{bmatrix}.
\]

Problem 7

Consider the matrix:

\[A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}\]

Diagonalize the matrix \( A \), and find the matrix power \( A^n \). Multiply out the matrix so that each element has a concrete expression.

Answer. Diagonalization:

\[AP = PD, \quad P = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\]

Matrix Power:

\[A^n = PD^n P^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \cdot 3^n + 3(-2)^n & 3^{n+1} - 3(-2)^n \\ 2 \cdot 3^n + (-2)^{n+1} & 3^{n+1} - (-2)^{n+1} \end{bmatrix}\]
Problem 8

Find the eigenvalues and *all* eigenvectors for the following matrices:

\[
\begin{bmatrix}
8 & -2 & -4 \\
3 & 1 & -2 \\
6 & -2 & -2
\end{bmatrix}, \quad
\begin{bmatrix}
-7 & -8 \\
18 & 17
\end{bmatrix}
\]

(We leave you to find the eigenvalues of the two-by-two matrix, but we tell you that the eigenvalues of the three-by-three matrix are 2, 2, 3, i.e., 2 is a double eigenvalue.) Then use the eigenvalues and eigenvectors to diagonalize the given matrices if possible. If not possible, explain why not.

Answer. Eigenvalues for the two-by-two: 5, 5, i.e., double root.

\[
\text{rref} \begin{bmatrix}
-12 & -8 \\
18 & 12
\end{bmatrix} = \begin{bmatrix}
1 & 2/3 \\
0 & 0
\end{bmatrix}
\]

There is just one nonpivot column, hence it takes one parameter to express all the solutions. The eigenvectors are \( t \begin{bmatrix} 2 \\ -3 \end{bmatrix} \) for \( t \neq 0 \). We can’t diagonalize because don’t have a two-dimensional eigenspace for the double eigenvalue.

Next we analyze the three-by-three. Since

\[
\text{rref} \begin{bmatrix}
5 & -2 & -4 \\
3 & -2 & -2 \\
6 & -2 & -5
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1/2 \\
0 & 0 & 0
\end{bmatrix},
\]

(note, just one nonpivot column) the eigenvectors for 3 are all vectors \( t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \) for \( t \neq 0 \). Since

\[
\text{rref} \begin{bmatrix}
6 & -2 & -4 \\
3 & -1 & -2 \\
6 & -2 & -4
\end{bmatrix} = \begin{bmatrix}
1 & -1/3 & -2/3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

(note two nonpivot columns) the eigenvectors for 2 are all vectors of the form

\[
s \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}
\]

where not both \( s \) and \( t \) are zero. We have enough eigenvectors so we get a diagonalization

\[
\begin{bmatrix}
8 & -2 & -4 \\
3 & 1 & -2 \\
6 & -2 & -2
\end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\]

As an extra precaution, besides multiplying out the matrices above to check, we also check that

\[
\text{det} \begin{bmatrix}
1 & 1 & 2 \\
2 & 3 & 0 \\
1 & 0 & 3
\end{bmatrix} = 3 \neq 0
\]

to make sure \( P \) is invertible.
Problem 9

Approximate the solution of the initial value problem
\[ y' = t - y^3/4, \quad y(1) = 1 \]
on the interval \( 1 \leq t \leq 1.4 \) by using Euler’s method with 2 steps.

Answer. Let \( F(t, y) = t - y^3/4. \) We have step-size \( h = (1.4 - 1)/2 = 0.2. \)

\[
  \begin{array}{ccc}
    k & t_k & y_k \\
    0 & 1 & 1 \\
    1 & 1.2 & 1.150 = 1 + F(1, 1) \cdot 0.2 \\
    2 & 1.4 & 1.314 = 1.150 + F(1.2, 1.150) \cdot 0.2 \\
  \end{array}
\]

Problem 10

Solve the following initial value problem:
\[ y'' + 6y' + 34y = 663 \cos(7t), \quad y(0) = 0, \quad y'(0) = 63 \]
Identify the steady-state and transient parts of the solution clearly. Express the steady-state part of the solution in the form \( A \cos(\omega t - \delta), \) where \( A > 0, \omega > 0, \delta \geq 0 \) and \( \delta < 2\pi. \) Numerical approximations to four decimal places of accuracy are requested for \( A, \omega \) and \( \delta. \)

Answer. 
\[
  y = -5 \cos(7t) + 14 \sin(7t) + 5e^{-3t} \cos(5t) - 4e^{-3t} \sin(5t) \\
  = 14.8660 \cos(7t - 1.9138) + 5e^{-3t} \cos(5t) - 4e^{-3t} \sin(5t)
\]

Problem 11

Find an initial value problem of the form
\[ y'' + by' + ky = A \cos(\omega t - \delta), \quad y(0) = y_0, \quad y'(0) = v_0 \]
of which
\[
  \cos(7t) - 3 \sin(7t) + 2e^{-3t} \cos(5t) - 3e^{-3t} \sin(5t)
\]
is the solution. Identify the stationary and transient parts of the solution, and determine the constants \( b, k, A, \omega, \delta, y_0 \) and \( v_0. \)

Answer. 
Stationary part:
\[ \cos(7t) - 3 \sin(7t). \]

Transient part:
\[ 2e^{-3t} \cos(5t) - 3e^{-3t} \sin(5t) \]
The characteristic roots are \( -3 \pm 5i \) given the form of the transient solution. From this, you can see that:
\[
  b = 6, \quad k = 34.
\]
Once this is done, you can find \( A, \omega \) and \( \delta \) as follows. Plug in the stationary part of the solution (which we call \( y_s \)) into the homogeneous equation:
\[
  y_s'' + 6y_s' + 34y_s = -141 \cos(7t) + 3 \sin(7t).
\]
This must be equal to the right hand side of the equation. Thus,

\[ \omega = 7, \ A = \sqrt{141^2 + 3^2} = 141.03, \]
\[ \tan(\delta) = -\frac{3}{141}, \ -\pi/2 < \delta < 0, \ \delta = -0.0212 \]

You can find the value of \( v_0 \) and \( y_0 \) easily from the solution.

\[ y_0 = 3, \ v_0 = -42. \]

**Problem 12**

Convert the initial value problem

\[ 3y'' + 24y' + 48y = 96 \cos(7t), \ y(0) = -5, \ y'(0) = 1 \]

to an initial value problem for a system of two linear first order differential equations. Make sure to specify the relationship of the variables in the converted IVP and to the variables in the given IVP.

**Answer.**

Let \( y = u \) and \( y' = w \). Solving for \( y'' \) we have \( y'' = -8y' - 16y + 32 \cos(7t) \).

\[
\begin{bmatrix}
u' \\
w'
\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -16 & -8\end{bmatrix} \begin{bmatrix}u \\
w\end{bmatrix} + \begin{bmatrix}0 \\
32 \cos(7t)\end{bmatrix}, \begin{bmatrix}u(0) \\
w(0)\end{bmatrix} = \begin{bmatrix}-5 \\
1\end{bmatrix}
\]

**Problem 13**

Are the vectors
\[
\begin{bmatrix}1 \\
2 \\
3 \\
4\end{bmatrix}, \begin{bmatrix}5 \\
6 \\
7 \\
8\end{bmatrix}, \begin{bmatrix}10 \\
11 \\
12 \\
13\end{bmatrix}
\]
linearly independent? If not, express one of the vectors as a linear combination of the others.

**Answer.**

No, the vectors are not linearly independent. This is inarguably established by the equation

\[
\begin{bmatrix}10 \\
11 \\
12 \\
13\end{bmatrix} = \frac{9}{4} \begin{bmatrix}5 \\
6 \\
7 \\
8\end{bmatrix} - \frac{5}{4} \begin{bmatrix}1 \\
2 \\
3 \\
4\end{bmatrix}.
\]
Problem 14

Express the vector \[
\begin{bmatrix}
-5 \\
3 \\
22 \\
5
\end{bmatrix}
\] as a linear combination of \[
\begin{bmatrix}
1 \\
1 \\
2 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
0 \\
3 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
-1 \\
1 \\
4 \\
1
\end{bmatrix}
\] if possible. If not possible, explain why. Then same question with \[
\begin{bmatrix}
-5 \\
3 \\
22 \\
5
\end{bmatrix}
\] replaced by \[
\begin{bmatrix}
-5 \\
3 \\
22 \\
4
\end{bmatrix}
\] (only a tiny change).

Answer.
\[
\begin{bmatrix}
-5 \\
3 \\
22 \\
5
\end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 1 \\ 4 \\ 1 \end{bmatrix}
\]

The “tiny change” case is impossible because \[
\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 22 \\ 4 \end{bmatrix}
\] are linearly independent. (Take, say, a determinant to prove this.)

Problem 15

Solve the following IVP’s:
\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -29 & 20 \\ -52 & 35 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 41 \end{bmatrix}
\]

\[
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & -3 & 7 \\ -2 & 9 & -1 \\ 14 & -3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}
\]

For the last IVP we tell you that the eigenvalues are \(-12, 6, 12\).

Answer.
\[
\begin{bmatrix} x \\ y \end{bmatrix} = -3e^{-4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 5e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = 6e^{3t} \left( \cos(4t) \begin{bmatrix} 5 \\ 8 \end{bmatrix} - \sin(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) - 7e^{3t} \left( \sin(4t) \begin{bmatrix} 5 \\ 8 \end{bmatrix} + \cos(4t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)
\]

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2e^{-12t} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + e^{6t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3e^{12t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\]