September 7, 2004; Due September 21, 2004.

Math 5652: Homework set #1: Review of probability (fall 2004)

1) Suppose X_1 is a Gaussian random variable with mean μ and variance $\sigma^2 > 0$, X_2 is a random variable distributed uniformly on $[a, b]$, X_3 is an exponential random variable of parameter μ, X_4 is a Poisson random variable of parameter λ, X_5 represents the number of successes in n independent trials with probability of success p each, $X_6 = X_1^2$, $X_7 = 3X_1 + 4$. Define further X_8 in the following way: flip a coin, independent of X_1, X_3, with $P(\text{head} = p)$. If the result is a head then $X_8 = X_1$, otherwise $X_8 = X_3$.

(1) Draw (no computer plots, just free hand-drawing with enough data to represent the functions involved) the cumulative distribution of each of the random variables X_i, and determine whether a probability density function exists. If it does, compute it.
(2) Compute the mean and variance of each random variable (you may quote from the book for standard random variables).
(3) Evaluate the third moment of X_2 and X_8.
(4) Compute the characteristic function $\phi_j(\lambda) := E(e^{i\lambda X_j})$ for the r.v.'s X_j.
(5) Evaluate $E[X_8|X_1]$.
(6) Evaluate the distribution function of X_8 conditioned on X_1.

2) Construct a probability space corresponding to 5 coin tosses of a biased coin with $P(\text{coin } j \text{ is head}) = p$, $j = 1, \ldots, 5$. Is the space you constructed the unique one possible?

3) Let (X_1, X_2, X_3) be jointly Gaussian of non-degenerate covariance matrix Λ_X and mean vector μ_X, that is with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$,

$$E(e^{i \sum_{j=1}^{3} \lambda_j X_j}) = e^{i\lambda \cdot \mu_X - \frac{1}{2}\lambda^T \Lambda_X \lambda} .$$

Compute $E(X_2|X_1)$ and $E(X_2|X_1, X_3)$. Find a deterministic affine transformation $Y = AX + B$ such that Y_i, $i = 1, 2, 3$ are independent and have variances $\sigma_i^2 > 0$. What is the law of Y_1 under this transformation?

4) Write an expression for the probability P_n to have $n/4$ successful outcomes out of n independent trials with probability of success of each being equal to p.
(* - not mandatory) Can you evaluate a good bound on

$$\frac{1}{n} \log P_n$$

as n gets large? even harder, can you compute the limit?

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 2 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.