September 21, 2004; Due October 5, 2004.

Math 5652: Homework set #2: Markov Chains (Fall 2004)

1) Let α be a deterministic constant, let Y_i be a sequence of i.i.d. Bernoulli(p) random variables, $p \in (0,1)$, and set $X_n = \alpha Y_n + Y_{n-1}$. For which values of α is $\{X_n\}$ a time-homogeneous Markov process? Given such α, determine the process's state space, its transition probabilities, and classify the states to transient or recurrent states. Find the stationary distribution of the process. *Remark: for values of α in which you declare the process not to be Markov, prove your claim!.*

2) Let $\{X_n\}$ be a finite state, homogeneous Markov chain, whose transition probability matrix $P(i,j)$ is doubly stochastic, that is $\sum_i P(i,j) = \sum_j P(i,j) = 1$, and such that $P(i,j) > 0$ for all i,j. Compute the stationary distribution for this chain. Is the stationary distribution unique? And what happens if the state space is not finite?

3) Repeat question 2, this time with a transition matrix satisfying the equation $q(i)P(i,j) = q(j)P(j,i)$ for some (positive) vector q.

4) There are m balls, labeled $1, 2, \ldots, m$, in two boxes, A and B. At step n, one chooses uniformly and independently of the history, an integer I_n in the set $\{1, \ldots, m\}$, and moves the I_n-ball from the box where it is to the other box. Let X_n denote the number of balls in the A box after n steps. Prove that X_n is a time-homogeneous Markov process, determine its transition probability matrix, classify its states, and find its stationary distribution when $m = 3$.

5) A slotted communication network divides time into slots. At the i-th slot, a number A_i of customers arrive. The sequence $\{A_i\}$ is i.i.d. and Poisson(λ). The service of a customer takes always 1 time-slots, and the server can serve at most one customer at a time. The queue length is defined as the difference between the total number of arrivals up to time n and the total number of services ended up to time n. Provide a Markov model of the system that allows you to answer the following two questions:

a) Find a range of parameters λ such that the limit $\lim_n P(Q_n \le x)$ exist and is nonzero.

b) Find a range of parameters λ such that the limit $\lim_n P(Q_n \le x)$ exist and is zero. How does a typical sample path of Q_n look in this situation?

(*) harder: can you classify those λ for which one of the above behaviors occur?

6) The following is a model of cell splitting: at time n, a living cell is split into two cells with probability p, or dies with probability $1-p$. The splitting of different cells is an independent sequence. Let Z_n be the number of living cells at time n. Assume $Z_0 = 2$. Prove that $\{Z_n\}$ is a time-homogeneous Markov process, determine its state space and its transition matrix, find its stationary distribution(s), and classify the states as transient or recurrent.

7) Let $\{X_n\}$ be a finite state time-homogeneous Markov chain, with state space $S = \{1, \ldots, N\}$. Suppose 1 is a recurrent state. Let τ_i be the successive visits to state 1, that is $\tau_1 = \min\{n > 0 : X_n = 1\}$ and $\tau_i = \min\{n > \tau_{i-1} : X_n = 1\}, i \ge 2$.

a) Set $Z_n = X_{\tau_n + 1}$. Is Z_n Markov? If yes, find its state space and classify its states.

b) Let $v(x) = E(\tau_1 | X_0 = x)$, for $x \in S$. Prove that $v(x) - \sum_{y \neq 1} P(x,y)v(y) = 1$.

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 2 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.