

Corrected version, September 26, 2002; Due October 4, 2002.

Math 8651: Homework set #3 (fall 2002)

1. Prove that every Lebesgue measurable subset of $[0, 1]$ with positive Lebesgue measure contains a non-measurable subset.
2. The ternary expansion of a point $x \in [0, 1]$ is obtained by writing $x = \sum a_i 3^{-i}$ where $a_i \in \{0, 1, 2\}$.
 - a) Prove that except for countably many points in $[0, 1]$, the ternary expansion is unique. Modify the definition to make it unique everywhere.
 - b) Prove that the set of points in $[0, 1]$ with $a_i \neq 1$ for all i coincides with the Cantor set \mathcal{C} , except possibly at countably many points.
 - c) Construct a probability measure P on the interval $[0, 1]$ equipped with the Borel σ -field, such that $P(\mathcal{C}) = 1$ but $P(\{x\}) = 0$ for any point $x \in [0, 1]$.
 - d) Is the measure you constructed in c) unique?
 - e) For $x \in [0, 1]$, define $N(x) = \infty$ if $x \in \mathcal{C}$, and otherwise define $N = \min\{n : a_n = 1\}$. Define

$$h(x) = \sum_{n=1}^{N(x)-1} 2^{-(n+1)} a_n + 2^{-N(x)}.$$

Prove that $h : R \rightarrow R$ is continuous, monotone, increases only on \mathcal{C} , and maps \mathcal{C} to $[0, 1]$.

f) Define $k(x) = (h(x) + x)/2$. Prove that k is invertible, that $g = k^{-1}$ is continuous, and that $k(\mathcal{C})$ is Lebesgue measurable and possesses positive Lebesgue measure.

g) Using Exercise 1, find a set $A \subset [0, 1]$ that is Lebesgue measurable but such that $k(A)$ is not Lebesgue measurable. *Note: you have just constructed a continuous function g that is not measurable as a function from (R, \mathcal{L}) to itself!*

3. Prove that Lebesgue's measure μ on $[0, 1]$ is *regular* in the sense that if A is a Lebesgue measurable set, then for each $\epsilon > 0$ there exist an open set O_ϵ and a closed set F_ϵ , such that $F_\epsilon \subset A \subset O_\epsilon$ and $\mu(O_\epsilon \setminus A) < \epsilon$, $\mu(A \setminus F_\epsilon) < \epsilon$.
4. Suppose (Ω, \mathcal{F}, P) is a complete probability space, and that $f : \Omega \rightarrow \Psi$ is a measurable map from (Ω, \mathcal{F}) to (ψ, \mathcal{G}) . Suppose that $g : \Omega \rightarrow \Psi$ satisfies that $f = g$, P -almost everywhere, that is $P(\{\omega : f(\omega) \neq g(\omega)\}) = 0$. Prove that g is a measurable map between the same spaces.
5. Suppose that f and g are measurable functions from (R, \mathcal{L}) to (R, \mathcal{B}) . Construct an example where $f \circ g : R \rightarrow R$ is *not* measurable as a function from (R, \mathcal{L}) to (R, \mathcal{B}) .

Hint: Set $f(x) = 1_A(x)$ and g as in Exercise 2.

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 3 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.