October 7, 2002; Due October 18, 2002. Corrected October 24, 2002.

Math 8651: Homework set #4 (fall 2002)

1. Let $X : [0,1]^2 \to R$ be measurable as a map from $([0,1]^2, \mathcal{B})$ to $(R, \mathcal{B})$. Prove that for any fixed $x \in [0,1]$, the function $Z(y) := X(x,y)$ is measurable as a map from $([0,1], \mathcal{B})$ to $(R, \mathcal{B})$.

2. Let X be a bounded random variable. Prove that
$$EX = \inf_{Z: Z \geq X, Z \text{ is simple}} EZ\,.$$

3. Suppose that $X : \Omega \mapsto R_+$ is bounded and such that
$$\inf_{Z: Z \geq X, Z \text{ is simple}} EZ = \sup_{Z: Z \leq X, Z \text{ is simple}} EZ\,.$$
Prove that X is measurable as a map from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B})$.

4. We have seen in class that if $X_n \to X$ and $|X_n| \leq Z$ with $EZ < \infty$ then $EX_n \to EX$.
Prove that the conclusion holds true if you assume that $X_n \to X$ almost everywhere instead of everywhere.

5. Fix $\Omega = [0,1]$ and set $\mathcal{F}$ to be the Lebesgue σ-field. Assume that $X : \Omega \to R$ is Riemann integrable. Prove that X is a random variable, and show that if P is the uniform measure on Ω then EX equals the Riemann integral of $X : [0,1] \to R$.
(Reminder: a function f is Riemann integrable if one may find step functions $f_n \leq f \leq g_n$ with $|\int f_n(x)dx - \int g_n(x)dx| \to 0$, with the obvious definition of the integral of a step function given by the sum of width of steps times their height).

6. Prove that if X is a non-negative real valued random variable with distribution function $F_X(t)$, then
$$E(X) = \int_0^\infty (1 - F_X(t))dt\,.$$

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 3 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.