December 2, 2002; Corrected December 6, 2002. Due December 11, 2002.

Math 8651: Homework set #8 (fall 2002)

1. Let μ_n be a sequence of probability distributions on R, such that for any bounded continuous function f of compact support it holds that

$$\int f d\mu_n \to_{n\to\infty} \int f d\mu \,.$$

   Does this imply the weak convergence of μ_n to μ?

2. Let X_n be a sequence of integer-valued random variables, with

$$P(X_n = k) \to_{n\to\infty} a_k \,.$$

   - Is the sequence of laws of the X_n's tight?
   - Assume now that $\sum a_k = 1$. Prove that the sequence of laws of X_n converges weakly to a distribution μ. Write μ explicitely.

3. Let X_i be i.i.d of zero mean and finite variance. Prove, using the CLT, that

$$\limsup \frac{\sum_{i=1}^n X_i}{\sqrt{n}} = \infty, \text{a.s.}\,.$$

4. *(*) (Here is the correction. If the question is too hard, solve replacing (1) below by the condition (2)).*

   Let X_i be a sequence of independent random variables, not identically distributed, with $EX_i = 0$, $\sup_i EX_i^2 < \infty$, and, for some $\beta > 2$,

$$\sup_i \frac{(EX_i^\beta)^{1/\beta}}{EX_i^2} < \infty. \tag{1}$$

   Assume

$$V_n := \sum_{i=1}^n EX_i^2 \to_{n\to\infty} \infty\,.$$

   Prove that $\sum_{i=1}^n X_i/\sqrt{V_n}$ converges in distribution to a standard Gaussian random variable. Find a counter example when the uniform boundedness assumption on EX_i^2 is dropped.

   "Baby problem": solve with (1) replaced by the condition that

$$\sup_i EX_i^2 < \infty, \inf_i EX_i^2 > 0\,. \tag{2}$$

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 3 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.