# Solutions: Math8651 - #6 (Fall 2002)

1. a. At the $n$-th step, the probability to pick a red ball is $p_n = 1/(n+1)$. Since $\sum p_n = \infty$, and since the events {pick a red ball in the n-th step} are independent by construction, the Borel-Cantelli Lemma yields that the probability of picking a red ball infinitely many times is 1.
   b. Now, $p_n = 1/(1+\sum_{i=1}^n i) = 1/(1+n(n+1)/2) \sim 2/n^2$. Since $\sum p_n < \infty$, the Borel-Cantelli Lemma now tells you that the probability of picking a red ball infinitely many times is 0.

2. Suppose that $X_n$ does not converge to $X$ in probability. Then there is an $\epsilon > 0$ and a sequence $n_k \to \infty$ such that

$$\liminf_{k\to\infty} P(|X_{n_k} - X| > \epsilon) > 0\,. \tag{1}$$

   By the data, there exists a further subsequence $n_{k_j}$ with $|X_{n_{k_j}} - X| \to 0$, almost surely. In particular, $|X_{n_{k_j}} - X| \to 0$, in probability. But this contradicts (1).

3. Since $f$ is continuous, for each $\epsilon > 0$ we can find a $\delta = \delta(\epsilon) > 0$ such that $|x_i - y_i| < \delta$, $i = 1, \ldots, m$ implies that $|f(x_1, \ldots, x_d) - f(y_1, \ldots, y_d)| < \epsilon$, uniformly on $y$ in a compact subset of $R^d$.

   Fix next an $\eta > 0$ and a compact subset $K_\eta \subset R^d$ such that $P((X_1, \ldots, X_d) \in K_\eta^c) < \eta$. Then,

$$\begin{aligned}
& P(|f(X_{1,n}, \ldots, X_{d,n}) - f(X_1, \ldots, X_d)| > \epsilon) \\
\leq\; & P((X_1, \ldots, X_d) \in K_\eta^c) + P(|(X_{1,n}, \ldots, X_{d,n}) - (X_1, \ldots, X_d)|_\infty > \delta(\epsilon)) \\
\leq\; & \eta + P(|(X_{1,n}, \ldots, X_{d,n}) - (X_1, \ldots, X_d)|_\infty > \delta(\epsilon)) \\
\to_{n\to\infty}\; & \eta\,.
\end{aligned}$$

   $\eta$ being arbitrary, the proof of the first past is completed.

   To see a counter example when $f$ is not continuous, take $f(x) = 1_{x\leq 0}$, $\Omega = [0,1]$, and $P$ the uniform measure (with the Lebesgue $\sigma$-field). Define $X_n = 1_{\omega<1/n}$. Then $X_n$ converges to 0 in probability (in fact, almost surely) but $f(X_n) = 0$ does not converge to $f(0) = 1$.

4. Write $Y_n = |X_n|$. Then

$$\sum P(Y_n > n) = \sum P(Y_1 > n) \geq \int_1^\infty P(Y_1 > t)dt \geq \int_0^\infty P(Y_1 > t)dt - 1 = EY_1 - 1 = \infty\,.$$

   Hence, because the $Y_i$'s are independent, $P(Y_n > n \text{ infinitely often}) = 1$. The conclusion follows by noting that each time $|X_n| > n$, one has $|S_n - S_{n-1}| \geq n$.

5. a. $P(2X_1 > x) = P(X_1 > x/2) = e^{-x/2}$.

b.

$$P(X_1+X_2 > x) = \int_0^\infty e^{-y} P(X_1 > x-y)dy = \int_0^x e^{-x} dy + \int_x^\infty e^{-y} dy = xe^{-x} + e^{-x}.$$

(The density thus is $xe^{-x}$, i.e. $X_1 + X_2$ is Gamma(2,1) distributed).