Solutions: Math8651 - #7 (Fall 2002)

1. Fix a constant A, and set $X_i^A = X_i 1_{|X_i| \le iA}$. Note that $1/nE(\max_{i<n} |X_i^A|) \le A$. On the other hand,

$$E(\max_{i \le n} |X_i|/n) \le n^{-1} E(\max_{i \le n} |X_i^A|) + n^{-1} E \sum_{i=1}^{n} E(|X_i - X_i^A|)\,.$$

Note that $E(|X_i - X_i^A|) = E(|X_1 - X_1^{iA}|) \to_{i\to\infty} 0$ by dominated convergence, and hence

$$\limsup E(\max_{i \le n} |X_i|/n) \le A\,.$$

A being arbitrary, the claim follows.

2. First, note that as a consequence of Fubini's theorem, and the fact that the probability distribution function is continuous,

$$P(X_i = X_j) = \int P(X_j = z) dP(z) = 0. \tag{1}$$

Next, using the exchangeability of the X_i's,

$$P(A_n) = P(X_n > \max_{k \ne n} X_k) = P(X_i > \max_{k \ne i} X_k) =: P(B_{i,n}))$$

while, by (1), $1 = \sum_{i=1}^{n} P(B_{i,n})$. Hence, $nP(A_n) = 1$.

Also, for any indices $i_1 < i_2 < i_k$, and $i_0 = 1$,

$$P(\cap_{j=1}^{k} A_{i_j}) = P(\cap_{j=1}^{k} \{X_{i_j} > \max_{i_{j-1} \le \ell < i_j} X_\ell\}) := P(\mathbf{X_i} \in B_k)$$

where $\mathbf{i} = (1, \ldots, i_k)$ and for any multi index $\mathbf{j} = (j_1, \ldots, j_{i_k})$, $\mathbf{X_j} = (X_{j_1}, X_{j_2}, \ldots, X_{j_{i_k}})$. Let π be a permutation of the indices $1, \ldots, i_k$. Then, by exchangeability, $P(\mathbf{X}_{\pi\mathbf{i}} \in B_k) = P(\mathbf{X_i} \in B_k)$.

Pick now a permutation at random, uniformly over the $i_k!$ permutations, and apply Fubini's theorem:

$$\frac{1}{i_k!} \sum_{\pi} P(\mathbf{X}_{\pi\mathbf{i}} \in B_k) = E(\frac{1}{i_k!} \sum_{\pi} 1_{\mathbf{X}_{\pi\mathbf{i}} \in B_k})\,. \tag{2}$$

So we need to count, for a given $(X_1, \ldots, X_{i_k})$ with no two entries equal, the number N_k of permutations that make

$$\mathbf{X}_{\pi\mathbf{i}} \in B_k. \tag{3}$$

Fix as π_0 the permutation such that $X_{\pi_0 1} < X_{\pi 2} < \ldots < X_{\pi_0 i_k}$. In order for a permutation π to satisfy (3), one has to have $\pi(i_k) = \pi_0(i_k)$, and one has to have that $(X_{\pi 1}, \ldots, X_{\pi i_{k-1}}) \in B_{k-1}$. There are $\begin{pmatrix} i_k - i_{k-1} - 1 \\ i_k - 1 \end{pmatrix}$

different ways to split the $i_k - 1$ indices into groups of size $i_k - i_{k-1} - 1$ and i_{k-1}. So the number of permutations needed is

$$N_k = (i_k - i_{k-1} - 1)! \binom{i_k - i_{k-1} - 1}{i_k - 1} N_{k-1} = \frac{(i_k - 1)!}{i_{k-1}!} N_{k-1}$$

Iterating, one gets

$$N_k = (i_k - 1)! \prod_{j=1}^{k-1} \frac{(i_j - 1)!}{i_j!} = \frac{(i_k - 1)!}{\prod_{j=1}^{k-1} i_j}$$

Substituting in (2), one gets

$$P(\mathbf{X_i} \in B_k) = \frac{1}{\prod_{j=1}^{k} i_j} = \prod_{j=1}^{k} P(A_{i_j})\,.$$

Claim: $P(A_i \,\text{infinitely often}) = 1$ This follows from the independence and the Borel-Cantelli lemma, for $\sum 1/i = \infty$.

Finally, $T = \sum_{j=2}^{\infty} j 1_{A_j \cap_{i=2}^{j-1} A_i^c}$ (where an intersection over an empty set of indices is taken as the whole space). Hence,

$$ET = \sum_{j=2}^{\infty} j P(A_j \cap_{i=2}^{j-1} A_i^c) = \sum_{j=2}^{\infty} \prod_{i=2}^{j-1} (1 - 1/i) \geq C \sum_j \frac{1}{j} = \infty\,.$$

3. Of course, one assumes $EX_1^2 > 0$. We did something similar in class, where we proved that $P(\limsup |S_n|/\sqrt{n} = \infty)$. If X_i are symmetric (that is, the law of X_1 is the same as the law of $-X_1$), this immediately implies $P(\limsup S_n/\sqrt{n} = \infty) \geq 1/2$, and then the $0-1$ law shows that this probability is actually 1. If the law of X_1 is not symmetric, take an independent copy of X_i, say X_i', and apply the conclusion to $Z_i = X_i - X_i'$.

4. Almost sure convergence of the X_i's implies a.s. convergence of the S_n/n, because for almost every ω and every $\epsilon > 0$ one may find an $n_0 = n_0(\omega, \epsilon)$ such that for $n > n_0$, $|X_n| < \epsilon$. Thus, $\limsup |S_n|/n < \epsilon$.

   Convergence in probability of the X_i does NOT imply the convergence in probability of S_n. A simple example is to take $X_i = i^2$ with probability $1/\log i$ (check that the event $X_i = i^2$ occurs infinitely many times!).

5. This is a simple application of the strong LLN: recall that the a_i are i.i.d., uniform on $\{0, 1, 2\}$. Hence, almost surely, since $Ea_1 = 1, E1_{a_1=0} = 1/3$,

$$\limsup n^{-1} \sum_{k=1}^{n} a_k = \lim n^{-1} \sum_{k=1}^{n} a_k = 1$$

   and

$$\limsup n^{-1} \sum_{k=1}^{n} 1_{a_k=0} = \lim n^{-1} \sum_{k=1}^{n} 1_{a_k=0} = 1/3.$$